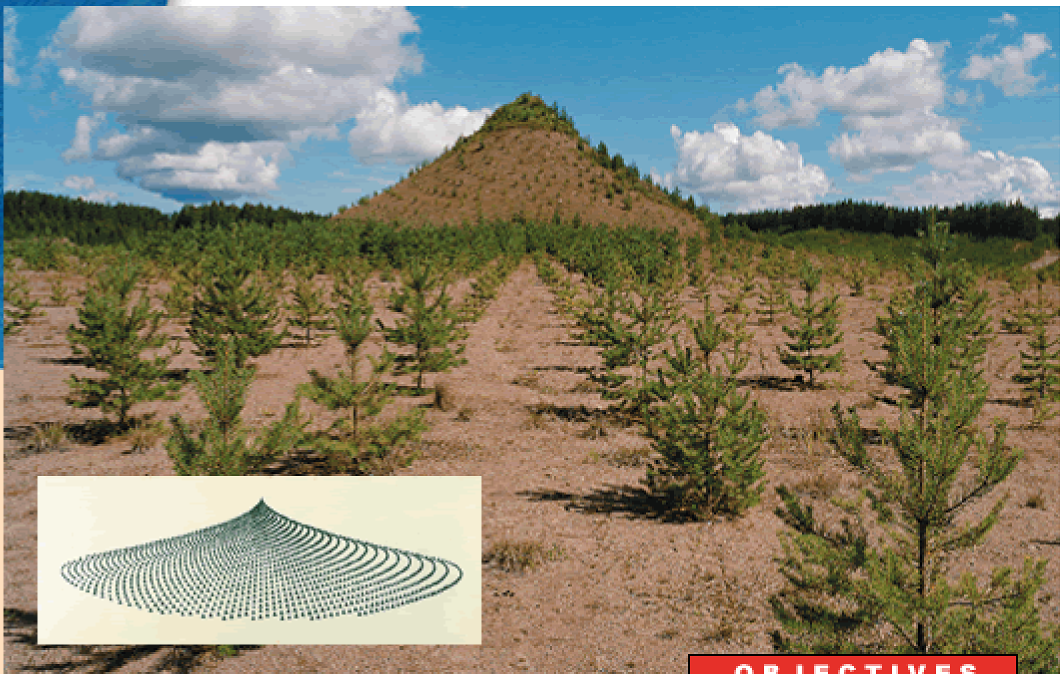
CHAPTER 5

Exponential, Power, and Logarithmic Functions



Art can take on living forms. To create *Tree Mountain,* conceptualized by artist Agnes Denes, 11,000 people planted 11,000 trees on a human-made mountain in Finland, on a former gravel pit. The trees were planted in a mathematical pattern similar to a golden spiral, imitating the arrangement of seeds on a sunflower. All living things grow and eventually decay. Both growth and decay can be modeled using exponential functions.

Tree Mountain-A Living Time Capsule-11,000 People, 11,000 Trees, 400 Years 1992-1996, Ylöjärvi, Finland, (420 x 270 x 28 meters) © Agnes Denes.

OBJECTIVES

In this chapter you will

- **write explicit equations for geometric sequences**
- **use exponential functions to model real-world growth and decay scenarios**
- **review the properties of exponents and the meaning of rational exponents**
- **learn how to find the inverse of a function**
- **apply logarithms, the inverses of exponential functions**

LESSON 5.1

Exponential Functions

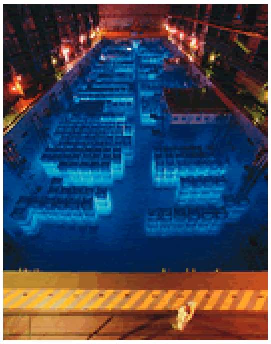
Life shrinks or expands in proportion to one 's courage.

ANAÏS NIN

In Chapter 1, you used sequences and recursive rules to model geometric growth or decay of money, populations, and other quantities. Recursive formulas generate only discrete values, such as the amount of money after one year or two years, or the population in a certain year. Usually growth and decay happen continuously. In this lesson you will focus on finding explicit formulas for these patterns, which will allow you to model situations involving continuous growth and decay, or to find discrete points without using recursion.

Science CONNECTION

Certain atoms are unstable—their nuclei can split apart, emitting radiation and resulting in a more stable atom. This process is called radioactive decay. The time it takes for half the atoms in a radioactive sample to decay is called **half-life,** and the half-life is specific to the element. For instance, the half-life of carbon-14 is 5730 years, whereas the half-life of uranium-238 is 4.5 billion years.



Submerged for two years in a storage tank in La Hague, France, this radioactive waste glows blue. The blue light is known as the "Cherenkov glow."

Investigation

Radioactive Decay

You will need

- one die per person

This investigation is a simulation of radioactive decay. Each person will need a standard six-sided die. [See Calculator Note 1L to simulate this with your calculator instead.] Each standing person represents a radioactive atom in a sample. The people who sit down at each stage represent the atoms that underwent radioactive decay.

Procedure Note

1. All members of the class should stand up, except for the recorder. The recorder counts and records the number standing at each stage.
2. Each standing person rolls a die, and anyone who gets a 1 sits down.
3. Wait for the recorder to count and record the number of people standing.
4. Repeat Steps 2 and 3 until fewer than three students are standing.

Step 1 Follow the Procedure Note to collect data in the form (*stage*, *number standing*).

Step 2 Graph your data and write a recursive formula that models it.

Step 3 Write an expression to calculate the 8th term using only u_0 and the ratio.

Step 4 Write an expression that calculates the nth term using only u_0 and the ratio.

Step 5 What was the half-life of this sample?

Step 6 Write a paragraph explaining how this investigation simulates the life of a radioactive sample.

In Chapter 3, you learned how to find an equation of the line that passes through points of an arithmetic sequence. In this lesson you will find an equation of the curve that passes through points of a geometric sequence.

You probably recognized the geometric decay model in the investigation. As you learned in Chapter 1, geometric decay is nonlinear. At each step the previous term is multiplied by a common ratio. So, the nth term contains the common ratio multiplied n times. You use exponents to represent a number that appears as a factor n times, so you use exponential functions to model geometric growth. An **exponential function** is a continuous function with a variable in the exponent, and it is used to model growth or decay.

EXAMPLE

Most automobiles depreciate as they get older. Suppose an automobile that originally costs $14,000 depreciates by one-fifth of its value every year.

a. What is the value of this automobile after two years?

b. When is this automobile worth half of its initial value?

Solution

a. The recursive formula gives automobile values only after one year, two years, and so on

The value decreases by $\frac{1}{5}$, or 0.2, each year, so to find the next term you continue to multiply by another $(1 - 0.2)$. You can use this fact to write an explicit formula.

$14000 \cdot (1 - 0.2)$	Value after 1 year.
$14000 \cdot (1 - 0.2) \cdot (1 - 0.2) = 14000(1 - 0.2)^2$	Value after 2 years.
$14000 \cdot (1 - 0.2) \cdot (1 - 0.2) \cdot (1 - 0.2) = 14000(1 - 0.2)^3$	Value after 3 years.
$14000 \cdot (1 - 0.2)^n$	Value after n years.

So, the explicit formula for automobile value is $u_n = 14000(1 - 0.2)^n$. The equation of the continuous function through the points of this sequence is

$$y = 14000(1 - 0.2)^x$$

You can use the continuous function to find the value of the car at any point. To find the value after $2\frac{1}{2}$ years, substitute 2.5 for x.

$$y = 14000(1 - 0.2)^{2.5} \approx \$8014.07$$

It makes sense that the automobile's value after $2\frac{1}{2}$ years should be between the values for u_2 and u_3, $8960 and $7168. Because this is not a linear function, finding the value halfway between 8960 and 7168 does not give an accurate value for the value of the car after $2\frac{1}{2}$ years.

b. To find when the automobile is worth half of its initial value, substitute 7000 for y and find x.

$y = 14000(1 - 0.2)^x$	Original equation.
$7000 = 14000(1 - 0.2)^x$	Substitute 7000 for y.
$0.5 = (1 - 0.2)^x$	Divide both sides by 14000.
$0.5 = (0.8)^x$	Combine like terms.

You don't yet know how to solve for x when x is an exponent, but you can experiment with different exponents to find one that produces a value close to 0.5. The value of $(0.8)^{3.106}$ is very close to 0.5. This means that the value of the car is about \$7000, or half of its original value, after 3.106 years (about 3 years 39 days). This is the half-life of the value of the automobile, or the amount of time needed for the value to decrease to half of the original amount.

Exponential Function

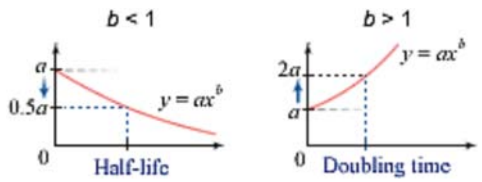
The general form of an exponential function is

$$y = ab^x$$

where the coefficient a is the y-intercept and the base b is the growth rate.

Exponential growth and decay are both modeled with the general form $y = ab^x$. Growth is modeled by a base that is greater than 1, and decay is modeled by a base that is less than 1. In general, a larger base models faster growth, and a base closer to 0 models faster decay.

All exponential growth curves have a **doubling time,** just as decay has a half-life. This time depends only on the ratio. For example, if the ratio is constant, it takes just as long to double \$1,000 to \$2,000 as it takes to double \$5,000 to \$10,000.



EXERCISES

You will need

Geometry software for Exercise **15**

Practice Your Skills

1. Evaluate each function at the given value.

a. $f(x) = 4.753(0.9421)^x$, $x = 5$ **b.** $g(h) = 238(1.37)^h$, $h = 14$

c. $h(t) = 47.3(0.835)^t + 22.3$, $t = 24$ **d.** $j(x) = 225(1.0825)^{x-3}$, $x = 37$

2. Record three terms of the sequence, and then write an explicit function for the sequence.

a. $u_0 = 16$
$u_n = 0.75u_{n-1}$ where $n \geq 1$

b. $u_0 = 24$
$u_n = 1.5u_{n-1}$ where $n \geq 1$

3. Evaluate each function at $x = 0$, $x = 1$, and $x = 2$, and then write a recursive formula for the pattern.

a. $f(x) = 125(0.6)^x$ **b.** $f(x) = 3(2)^x$

4. Calculate the ratio of the second term to the first term, and express the answer as a decimal value. State the percent increase or decrease.

a. 48, 36 **b.** 54, 72 **c.** 50, 47 **d.** 47, 50

Reason and Apply

5. In 1991, the population of the People's Republic of China was 1.151 billion, with a growth rate of 1.5% annually.

a. Write a recursive formula that models this growth. Let u_0 represent the population in 1991.

b. Complete a table recording the population for the years 1991 to 2000.

c. Define the variables and write an exponential equation that models this growth. Choose two data points from the table and show that your equation works.

d. The actual population of China in 2001 was 1.273 billion. How does this compare with the value predicted by your equation? What does this tell you?

Cultural CONNECTION

In 2001, the population of China was 1.273 billion. In large cities like Beijing and Shanghai, more than half of the people use bicycles as transportation. What would happen if half a billion Chinese bicyclists switched to cars? What if half of the drivers in U.S. cities switched to bicycles?

6. Jack planted a mysterious bean just outside his kitchen window. It immediately sprouted 2.56 cm above the ground. Jack kept a careful log of the plant's growth. He measured the height of the plant each day at 8:00 A.M. and recorded these data.

Day	0	1	2	3	4
Height (cm)	2.56	6.4	16	40	100

Kudzu, shown here in Oxford, Mississippi, is a fast-growing weed that was brought from Japan and once promoted for its erosion control. Today kudzu covers more than 7 million U.S. acres and spreads across about 120,000 more each year.

a. Define variables and write an exponential equation for this pattern. If the pattern were to continue, what would be the heights on the fifth and sixth days?

b. Jack's younger brother measured the plant at 8:00 P.M. on the evening of the third day and found it to be about 63.25 cm tall. Show how to find this value mathematically. (You may need to experiment with your calculator.)

c. Find the height of the sprout at 12:00 noon on the sixth day.

d. Find the doubling time for this plant.

e. Experiment with the equation to find the day and time (to the nearest hour) when the plant reaches a height of 1 km.

7. ***Mini-Investigation*** For 7a–d, graph the equations on your calculator.

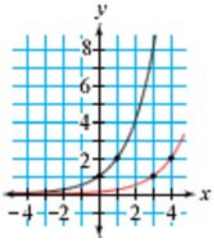
a. $y = 1.5^x$ **b.** $y = 2^x$ **c.** $y = 3^x$ **d.** $y = 4^x$

e. How do the graphs compare? What points (if any) do they have in common?

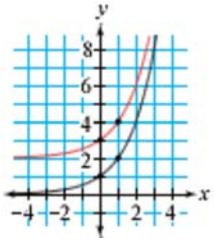
f. Predict what the graph of $y = 6^x$ will look like. Verify your prediction by using your calculator.

8. Each of the red curves is a transformation of the graph of $y = 2^x$, shown in black. Write an equation for each red curve.

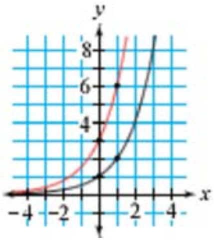
a.



b.



c.



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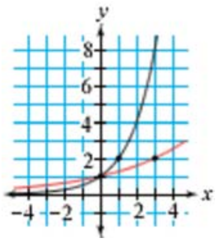


9. ***Mini-Investigation*** For 9a–d, graph the equations on your calculator.

a. $y = 0.2^x$

b. $y = 0.3^x$

c. $y = 0.5^x$

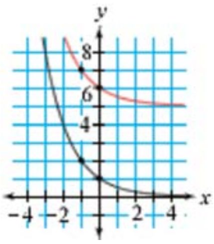
d. $y = 0.8^x$

e. How do the graphs compare? What points (if any) do they have in common?

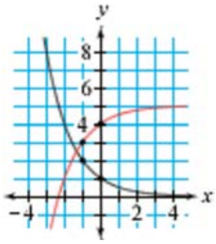
f. Predict what the graph of $y = 0.1^x$ will look like. Verify your prediction by using your calculator.

10. Each of the red curves is a transformation of the graph of $y = 0.5^x$, shown in black. Write an equation for each red curve.

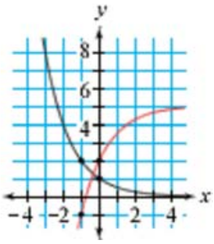
a.



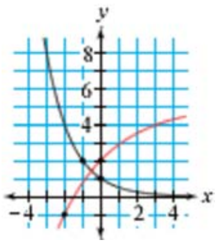
b.



c.



d.



11. The general form of an exponential equation, $y = ab^x$, is convenient when you know the y-intercept. Start with $f(0) = 30$ and $f(1) = 27$.

a. Find the common ratio.

b. Write the function $f(x)$ that passes through the two data points.

c. Graph $f(x)$ and $g(x) = f(x - 4)$ on the same axes.

d. What is the value of $g(4)$?

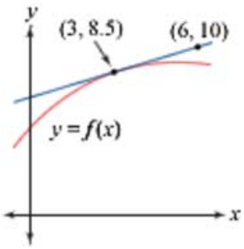
e. Write an equation for $g(x)$ that does not use its y-intercept.

f. Explain in your own words why $y = y_1 \cdot b^{x - x_1}$ might be called the point-ratio form.

This painting, *Inspiration Point* (2000), by contemporary American artist Nina Bovasso, implies an explosion of exponential growth.

Review

12. The graph shows a line and the graph of $y = f(x)$.



a. Complete the missing values to make a true statement.

$f(\underline{?}) = \underline{?}$.

b. Find the equation of the pictured line.

13. Janell starts 10 m from a motion sensor and walks at 2 m/s toward the sensor. When she is 3 m from the sensor, she stantly turns around and walks at the same speed back to the starting point.

a. Sketch a graph of the function that models Janell's walk.

b. Give the domain and range of the function.

c. Write an equation of the function.

14. ***Mini-Investigation*** Make up two linear functions, f and g. Enter Y1 = $f(x)$ and Y2 = $g(x)$. Enter $f(g(x))$ in Y3 as Y1(Y2) and $g(f(x))$ in Y4 as Y2(Y1).

a. Display the graphs of $f(g(x))$ and $g(f(x))$. Describe the relationship between them.

b. Change $f(x)$ or $g(x)$ or both, and again graph $f(g(x))$ and $g(f(x))$. Does the relationship you found in 14a still seem to be true?

c. Explain the relationship algebraically.

15. ***Technology*** Use geometry software to construct a circle. Label it circle M and measure its area.

a. Construct another circle with twice the area of circle M and label it circle L.

b. Construct another circle with half the area of circle M and label it circle S.

c. Describe the method you used to determine the size of each circle.

d. Calculate the ratio of the diameters of circle L to circle M, circle M to circle S, and circle L to circle S. Explain why these ratios make sense.

16. You can use different techniques to find the product of two binomials, such as $(x - 4)(x + 6)$.

a. Use a rectangle diagram to find the product.

b. You can use the distributive property to rewrite the expression $(x - 4)(x + 6)$ as $x(x + 6) - 4(x + 6)$. Use the distributive property again to find all the terms. Combine like terms.

c. Compare your answers to 16a and 16b. Are they the same?

d. Compare the methods in 16a and 16b. How are they alike?

Project

THE COST OF LIVING

You have probably heard someone say something like, "I remember when a hamburger cost five cents!" Did you know that the cost of living tends to increase exponentially? Talk to a relative or an acquaintance to see whether he or she remembers the cost of a specific item in a certain year. The item could be a meal, a movie ticket, a house in your neighborhood, or anything else the person recalls.

Research the current cost of that same item. How much has it increased? Use your two data points to write an exponential equation for the cost of the item.

When can you expect the cost of the item to be double what it is now?

Research the cost of the item in a different year. How close is this third data point to the value predicted by your model?

Your project should include

- Your data and sources.
- Your equation and why you chose that model.
- The doubling time for the cost of your item.
- An analysis of how well your model predicted the third data point.
- An analysis of how accurate you think your model is.

In 1944 about 10,000 fans formed a line stretching several blocks outside of Manhattan's Paramount Theater to see Frank Sinatra. About 100 police officers were called out to maintain order. Although the cost of seeing a live concert has changed since 1944, the enthusiasm of fans has not.

LESSON 5.2

Properties of Exponents and Power Functions

Frequently, you will need to rewrite a mathematical expression in a different form to make the expression easier to understand or an equation easier to solve. Recall that in an exponential expression, such as 4^3, the number 4 is called the **base** and the number 3 is called the **exponent.** You say that 4 is **raised to the power** of 3. If the exponent is a positive integer, you can write the expression in **expanded form,** for example, $4^3 = 4 \cdot 4 \cdot 4$. Because 4^3 equals 64, you say that 64 is a power of 4.

Investigation
Properties of Exponents

Use expanded form to review and generalize the properties of exponents.

Step 1 Write each product in expanded form, and then rewrite it in exponential form.

a. $2^3 \cdot 2^4$ **b.** $x^5 \cdot x^{12}$ **c.** $10^2 \cdot 10^5$

Step 2 Generalize your results from Step 1. $a^m \cdot a^n = \underline{?}$

Step 3 Write the numerator and denominator of each quotient in expanded form. Reduce by eliminating common factors, and then rewrite the factors that remain in exponential form.

a. $\frac{4^5}{4^2}$ **b.** $\frac{x^8}{x^6}$ **c.** $\frac{(0.94)^{15}}{(0.94)^5}$

Step 4 Generalize your results from Step 3. $\frac{a^m}{a^n} = \underline{?}$

Step 5 Write each quotient in expanded form, reduce, and rewrite in exponential form.

a. $\frac{2^3}{2^4}$ **b.** $\frac{4^5}{4^7}$ **c.** $\frac{x^3}{x^8}$

Step 6 Rewrite each quotient in Step 5 using the property you discovered in Step 4.

Step 7 Generalize your results from Steps 5 and 6. $\frac{1}{a^n} = \underline{?}$

Step 8 Write several expressions in the form $(a^n)^m$. Expand each, and then rewrite the expression in exponential form. Generalize your results.

Step 9 Write several expressions in the form $(a \cdot b)^n$. Don't multiply a times b. Expand each, and rewrite the expression in exponential form. Generalize your results.

Step 10 Show that $a^0 = 1$, using the properties you have discovered. Write at least two exponential expressions to support your explanation.

Here's a summary of the properties of exponents. You discovered some of these in the investigation. Try to write an example of each property.

For $a > 0$, $b > 0$, and all values of m and n, these properties are true:

Product Property of Exponents

$$a^m \cdot a^n = a^{m+n}$$

Quotient Property of Exponents

$$\frac{a^m}{a^n} = a^{m-n}$$

Definition of Negative Exponents

$$a^{-n} = \frac{1}{a^n} \text{ or } \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Zero Exponents

$$a^0 = 1$$

Power of a Power Property

$$(a^m)^n = a^{mn}$$

Power of a Product Property

$$(ab)^m = a^m b^m$$

Power of a Quotient Property

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Power Property of Equality

If $a = b$, then $a^n = b^n$.

Common Base Property of Equality

If $a^n = a^m$, then $n = m$.

In Lesson 5.1, you learned to solve equations that have a variable in the exponent by using a calculator to try various values for x. The properties of exponents allow you to solve these types of equations algebraically. One special case is when you can rewrite both sides of the equation with a common base. This strategy is fundamental to solving some of the equations you'll see later in this chapter.

EXAMPLE A

Solve.

a. $8^x = 4$

b. $27^x = \frac{1}{81}$

c. $\left(\frac{49}{9}\right)^x = \left(\frac{3}{7}\right)^{3/2}$

▶ Solution

If you use the power of a power property to convert each side of the equation to a common base, then you can solve without a calculator.

a. $8^x = 4$ — Original equation.

$(2^3)^x = 2^2$ — $8 = 2^3$ and $4 = 2^2$.

$2^{3x} = 2^2$ — Use the power of a power property to rewrite $(2^3)^x$ as 2^{3x}.

$3x = 2$ — Use the common base property of equality.

$x = \frac{2}{3}$ — Divide.

b. $27^x = \frac{1}{81}$ — Original equation.

$(3^3)^x = \frac{1}{3^4}$ — $27 = 3^3$ and $81 = 3^4$.

$3^{3x} = 3^{-4}$ — Use the power of a power property and the definition of negative exponents.

$3x = -4$ — Use the common base property of equality.

$x = -\frac{4}{3}$ — Divide.

c. $\left(\frac{49}{9}\right)^x = \left(\frac{3}{7}\right)^{3/2}$ — Original equation.

$\left(\frac{7^2}{3^2}\right)^x = \left(\frac{3}{7}\right)^{3/2}$ — $49 = 7^2$ and $9 = 3^2$.

$\left(\left(\frac{7}{3}\right)^2\right)^x = \left(\left(\frac{7}{3}\right)^{-1}\right)^{3/2}$ — Use the power of a quotient property and the definition of negative exponents.

$\left(\frac{7}{3}\right)^{2x} = \left(\frac{7}{3}\right)^{-3/2}$ — Use the power of a power property.

$2x = -\frac{3}{2}$ — Use the common base property of equality.

$x = -\frac{3}{4}$ — Divide.

Remember, it's always a good idea to check your answer with a calculator.
[See Calculator Note 5A to find out how to calculate roots and powers.]

An exponential function in the general form $y = ab^x$ has a variable as the exponent. A **power function,** in contrast, has a variable as the base.

Power Function

The general form of a power function is

$$y = ax^n$$

where a and n are constants.

You use different methods to solve power equations than to solve exponential equations. You must learn to recognize the difference between the two.

EXAMPLE B

Solve.

a. $x^4 = 3000$

b. $6x^{2.5} = 90$

Solution

To solve a power equation, use the power of a power property and choose an exponent that will undo the exponent on x.

a.	$x^4 = 3000$	Original equation.
	$(x^4)^{1/4} = 3000^{1/4}$	Use the power of a power property. Raising both sides to the power of $\frac{1}{4}$ "undoes" the power of 4 on x.
	$x \approx 7.40$	Use your calculator to approximate the value of $3000^{1/4}$.
b.	$6x^{2.5} = 90$	Original equation.
	$x^{2.5} = 15$	Divide both sides by 6.
	$(x^{2.5})^{1/2.5} = 15^{1/2.5}$	Use the power of a power property and choose the exponent $\frac{1}{2.5}$.
	$x \approx 2.95$	Approximate the value of $15^{1/2.5}$.

Solving equations symbolically is often no more than "undoing" the order of operations. So to solve $6x^{2.5} = 90$ you divide by 6 and then raise the result to the power of $\frac{1}{2.5}$.

The properties of exponents are defined only for positive bases. So, using these properties gives only one solution to each equation. In part a above, is $x \approx -7.40$ also a solution?

EXERCISES

One of the authors of this book, Ellen Kamischke, works with two students in Interlochen, Michigan.

Practice Your Skills

1. Rewrite each expression as a fraction without exponents. Verify that your answer is equivalent to the original expression by evaluating each on your calculator.

a. 5^{-3} **b.** -6^2

c. -3^{-4} **d.** $(-12)^{-2}$

e. $\left(\frac{3}{4}\right)^{-2}$ **f.** $\left(\frac{2}{7}\right)^{-1}$

2. Rewrite each expression in the form a^n.

a. $a^8 \cdot a^{-3}$ **b.** $\frac{b^6}{b^2}$ **c.** $(c^4)^5$ **d.** $\frac{d^0}{e^{-3}}$

3. State whether each equation is true or false. If it is false, explain why.

a. $3^5 \cdot 4^2 = 12^7$ **b.** $100(1.06)^x = 106^x$

c. $\frac{4^x}{4} = 1^x$ **d.** $\frac{6.6 \cdot 10^{12}}{8.8 \cdot 10^{-4}} = 7.5 \cdot 10^{15}$

4. Solve.

a. $3x = \frac{1}{9}$ **b.** $\left(\frac{5}{3}\right)^x = \frac{27}{125}$ **c.** $\left(\frac{1}{3}\right)^x = 243$ **d.** $5 \cdot 3^x = 5$

5. Solve each equation. If answers are not exact, approximate to two decimal places.

a. $x^7 = 4000$ **b.** $x^{0.5} = 28$ **c.** $x^{-3} = 247$

d. $5x^{1/4} + 6 = 10.2$ **e.** $3x^{-2} = 2x^4$ **f.** $-3x^{1/2} + (4x)^{1/2} = -1$

Reason and Apply

6. Rewrite each expression in the form ax^n.

a. $x^6 \cdot x^6$ **b.** $4x^6 \cdot 2x^6$ **c.** $(-5x^3)\cdot(-2x^4)$

d. $\frac{72x^7}{6x^2}$ **e.** $\left(\frac{6x^5}{3x}\right)^3$ **f.** $\left(\frac{20x^7}{4x}\right)^{-2}$

7. ***Mini-Investigation*** You've seen that the power of a product property allows you to rewrite $(a \cdot b)^n$ as $a^n \cdot b^n$. Is there a power of a sum property that allows you to rewrite $(a + b)^n$ as $a^n + b^n$? Write some numerical expressions in the form $(a + b)^n$ and evaluate them. Are your answers equivalent to $a^n + b^n$ always, sometimes, or never? Write a short paragraph that summarizes your findings.

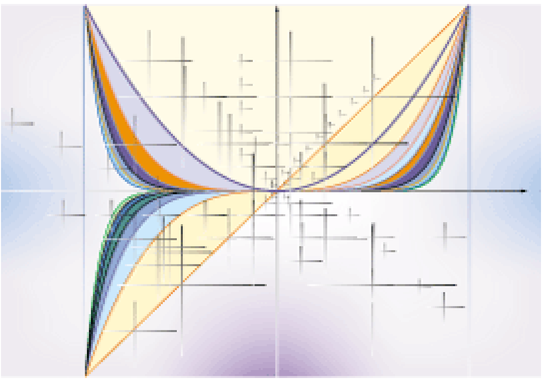
8. Consider this sequence:

$7^2, 7^{2.25}, 7^{2.5}, 7^{2.75}, 7^3$

a. Use your calculator to evaluate each term in the sequence. If answers are not exact, approximate to four decimal places.
b. Find the differences between the consecutive terms of the sequence. What do these differences tell you?
c. Find the ratios of the consecutive terms in 8a. What do these values tell you?
d. What observation can you make about these decimal powers?

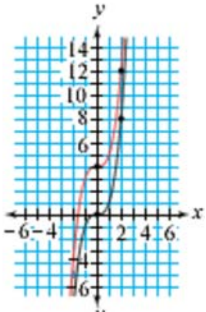
9. ***Mini-Investigation*** For 9a–d, graph the equations on your calculator.

a. $y = x^2$ **b.** $y = x^3$
c. $y = x^4$ **d.** $y = x^5$
e. How do the graphs compare? How do they contrast? What points (if any) do they have in common?
f. Predict what the graph of $y = x^6$ will look like. Verify your prediction by using your calculator.
g. Predict what the graph of $y = x^6$ will look like. Verify your prediction by using your calculator.



10. Each of the red curves is a transformation of the graph of $y = x^3$, shown in black. Write an equation for each red curve.

a.



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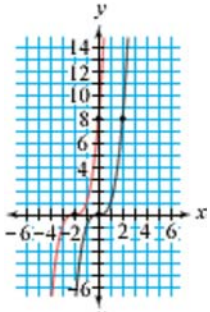

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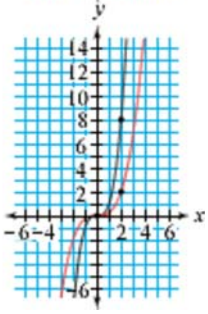
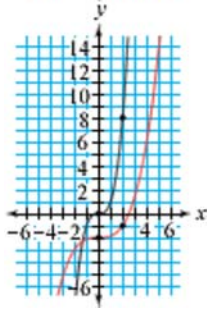
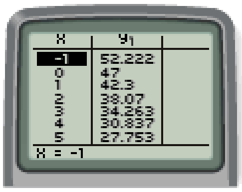


d.



11. Consider the exponential equation $y = 47(0.9)^x$. Several points satisfying the equation are shown in the calculator table. Notice that when $x = 0$, $y = 47$.



a. The expression $47(0.9)^x$ could be rewritten as $47(0.9)(0.9)^{x-1}$. Explain why this is true. Rewrite $47(0.9)(0.9)^{x-1}$ in the form $a \cdot b^{x-1}$.

b. The expression $47(0.9)^x$ could also be rewritten as $47(0.9)(0.9)(0.9)^{x-2}$. Rewrite $47(0.9)(0.9)(0.9)^{x-2}$ in the form $a \cdot b^{x-2}$.

c. Look for a connection between your answers to 11a and b, and the values in the table. State a conjecture or general equation that generalizes your findings.

12. A ball rebounds to a height of 30.0 cm on the third bounce and to a height of 5.2 cm on the sixth bounce.

a. Write two different yet equivalent equations in point-ratio form, $y = y_1 \cdot b^{x-x_1}$, using r for the ratio. Let x represent the bounce number, and let y represent the rebound height in centimeters.

b. Set the two equations equal to each other. Solve for r.

c. What height was the ball dropped from?

13. Solve.

a. $(x-3)^3 = 64$

b. $256^x = \frac{1}{16}$

c. $\frac{(x+5)^3}{(x+5)} = x^2 + 25$

14. **APPLICATION** A radioactive sample was created in 1980. In 2002, a technician measures the radioactivity at 42.0 rads. One year later, the radioactivity is 39.8 rads.

a. Find the ratio of radioactivity between 2002 and 2003. Approximate your answer to four decimal places.

b. Let x represent the year, and let y represent the radioactivity in rads. Write an equation in point-ratio form, $y = y_1 \cdot b^{x-x_1}$, using the point $(x_1, y_1) = (2002, 42)$.

c. Write an equation in point-ratio form using the point (2003, 39.8).

d. Calculate the radioactivity in 1980 using both equations.

e. Calculate the radioactivity in 2010 using both equations.

f. Use the properties of exponents to show that the equations in 14b and c are equivalent.

Review

15. Name the x-value that makes each equation true.

a. $37000000 = 3.7 \cdot 10^x$

b. $0.000801 = 8.01 \cdot 10^x$

b. $47500 = 4.75 \cdot 10^x$

d. $0.0461 = x \cdot 10^{-2}$

16. Solve this equation for y. Then carefully graph it on your paper.

$$\frac{y+3}{2} = (x+4)^2$$

17. Paul collects these time-distance data for a remote-controlled car.

Time (s)	5	8	8	10	15	18	22	24	31	32
Distance (m)	0.8	1.7	1.6	1.9	3.3	3.4	4.1	4.6	6.4	6.2

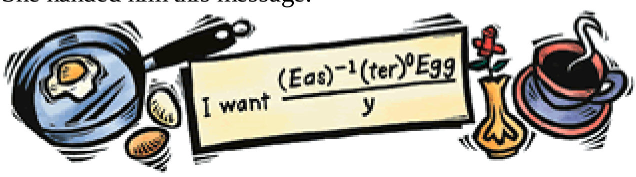
a. Define variables and make a scatter plot of these data.

b. Use the median-median line to estimate the car's speed. (*Note:* Don't do more work than necessary.)

IMPROVING YOUR REASONING SKILLS

Breakfast Is Served

Mr. Higgins told his wife, the mathematics professor, that he would make her breakfast. She handed him this message:



What should Mr. Higgins fix his wife for breakfast?

LESSON 5.3

Rational Exponents and Roots

In this lesson you will investigate properties of fractional, or **rational**, exponents. You will see how they can be useful in solving exponential and power equations and in finding an exponential curve to model data.

The volume and surface area of a cube, such as this fountain in Osaka, Japan, are related by rational exponents.

Investigation
Getting to the Root

In this investigation you'll explore the relationship between x and $x^{1/2}$ and learn how to find the values of some expressions with rational exponents.

Step 1 Use your calculator to create a table for $y = x^{1/2}$ at integer values of x. When is $x^{1/2}$ a positive integer? Describe the relationship between x and $x^{1/2}$.

Step 2 Graph $Y_1 = x^{1/2}$ in a friendly window with a factor of 2. This graph should look familiar to you. Make a conjecture about what other function is equivalent to $y = x^{1/2}$, enter your guess in Y_2, and verify that the equations give the same y-value at each x-value.

Step 3 State what you have discovered about raising a number to a power of $\frac{1}{2}$. Include an example with your statement.

Step 4 Clear the previous functions, and make a table for $y = 25^x$ with x incrementing by $\frac{1}{2}$.

Step 5 Study your table and explain any relationships you see. How could you find the value of $49^{3/2}$ without your calculator? Check your answer using the calculator.

Step 6 How could you find the value of $27^{2/3}$ without a calculator? Verify your response, and then test your strategy on $8^{5/3}$. Check your answer.

Step 7 Describe what it means to raise a number to a rational exponent, and generalize a procedure for simplifying $a^{m/n}$.

Rational exponents with a numerator of 1 indicate roots. For example, $x^{1/5}$ is the same as $\sqrt[5]{x}$, or the "fifth root of x," and $x^{1/n}$ is the same as $\sqrt[n]{x}$, or the "nth root of x." Recall that the fifth root of x is the number that, raised to the power of 5, gives you x. For rational exponents with numerators other than 1, such as $9^{3/2}$, the numerator is interpreted as the power to which to raise the root. That is, $9^{3/2}$ is the same as $(9^{1/2})^3$, or $\left(\sqrt{9}\right)^3$.

Definition of Rational Exponents

The power of a power property shows that $a^{m/n} = \left(a^{1/n}\right)^m$ and $a^{m/n} = a^{m/n} = (a^m)^{1/n}$, so

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \text{ or } \sqrt[n]{a^m} \text{ for } a > 0$$

Properties of rational exponents are useful in solving equations with exponents.

EXAMPLE A

Rewrite with rational exponents, and solve.

a. $\sqrt[4]{a} = 14$

b. $\sqrt[9]{b^5} = 26$

c. $\left(\sqrt[3]{c}\right)^8 = 47$

▶ Solution

Rewrite each expression with a rational exponent, then use properties of exponents to solve.

a.

$\sqrt[4]{a} = 14$	Original equation.
$a^{1/4} = 14$	Rewrite $\sqrt[4]{a}$ as $a^{1/4}$.
$\left(a^{1/4}\right)^4 = 14^4$	Raise both sides to the power of 4.
$a = 38416$	Evaluate 14^4.

b.

$\sqrt[9]{b^5} = 26$	Original equation.
$b^{5/9} = 26$	Rewrite as $\sqrt[9]{b^5}$ as $b^{5/9}$.
$\left(b^{5/9}\right)^{9/5} = 26^{9/5}$	Raise both sides to the power of $\frac{9}{5}$.
$b \approx 352.33$	Approximate the value of $26^{9/5}$.

c.

$\left(\sqrt[3]{c}\right)^8 = 47$	Original equation.
$c^{8/3} = 47$	Rewrite $\left(\sqrt[3]{c}\right)^8$ as $c^{8/3}$.
$\left(c^{8/3}\right)^{3/8} = 47^{3/8}$	Raise both sides to the power of $\frac{3}{8}$.
$c \approx 4.237$	Approximate the value of $47^{3/8}$.

Recall that properties of exponents give only one solution to an equation, because they are only defined for positive bases. Will negative values of *a*, *b*, or *c* work in any of the equations in Example A?

In the previous lesson, you learned that functions in the general form $y = ax^n$ are power functions. A rational function, such as $y = \sqrt[9]{x^5}$, is considered to be a power function because it can be rewritten as $y = x^{5/9}$. All the transformations you discovered for parabolas and square root curves also apply to any function that can be written in the general form $y = ax^n$.

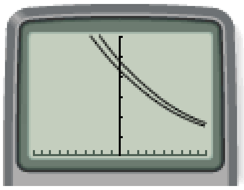
Recall that the equation of a line can be written using the point-slope form if you know a point on the line and the slope between points. Similarly, the equation for an exponential curve can be written if you know a point on the curve and the common ratio between points that are 1 horizontal unit apart. The **point-ratio form** of an exponential curve is $y = y_1 \cdot b^{x - x_1}$ where (x_1, y_1) is a point on the line and *b* is the ratio between points.

Point-Ratio Form

If an exponential curve passes through the point (x_1, y_1) and the function values have ratio b, the point-ratio form of the equation is

$$y = y_1 \cdot b^{x - x_1}$$

You have seen that if $x = 0$, then $y = a$ in the general exponential equation $y = a \cdot b^x$. This means that a is the initial value of the function at time 0 (the y-intercept) and b is the growth or decay ratio. This is consistent with the point-ratio form because when you substitute the point $(0, a)$ into the equation, you get $y = a \cdot b^{x-0}$, or $y = a \cdot b^x$.



The table and graph above show the exponential functions $Y1 = f(x) = 47(0.9)^x$ and $Y2 = g(x) = 42.3(0.9)^x$. Both the table and graph indicate that if the graph of function g is translated right 1 unit, it becomes the same as the graph of the function f. So $f(x) = g(x - 1)$, or $f(x) = 42.3(0.9)^{(x-1)} = 47(0.9)^x$. This shows that using the point (1, 42.3) in the point-ratio form gives you an equation equivalent to $y = a \cdot b^x$.

Try substituting another point, (x_1, y_1), along with the ratio $b = 0.9$ into the point-ratio form to convince yourself that any point (x_1, y_1) on the curve can be used to write an equation, $y = y_1 \cdot b^{x - x_1}$, that is equivalent to $y = a \cdot b^x$. You may want to use your graphing calculator or algebraic techniques.

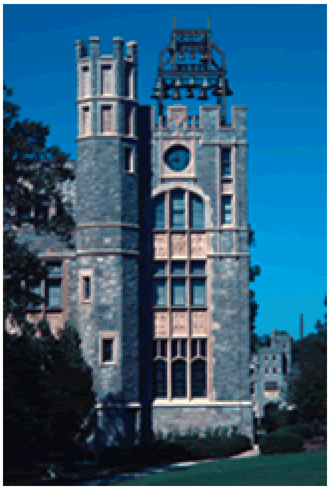
EXAMPLE B

Casey hit the bell in the school clock tower. Her pressure reader, held nearby, measured the sound intensity, or loudness, at 40 lb/in.^{2} after 4 s had elapsed and at 4.7 lb/in.^{2} after 7 s had elapsed. She remembers from her science class that sound decays exponentially.

a. Name two points that the exponential curve must pass through.

b. Find an exponential equation that models these data.

c. How loud was the bell when it was struck (at 0 s)?



The bell tower at Oglethorpe University in Atlanta, Georgia.

Solution

a. Time is the independent variable, *x*, and loudness is the dependent variable, *y*, so the two points are (4, 40) and (7, 4.7).

b. Start by substituting the coordinates of each of the two points into the point-ratio form, $y = y_1 \cdot b^{x - x_1}$.

$y = 40b^{x-4}$ and $y = 4.7b^{x-7}$

Note that you don't yet know what *b* is. If you were given *y*-values for two consecutive integer points, you could divide to find the ratio. In this case, however, there are 3 horizontal units between the two points you are given, so you'll need to solve for *b*.

$40b^{x-4} = 4.7b^{x-7}$	Use substitution to combine the two equations.
$b^{x-4} = \dfrac{4.7b^{x-7}}{40}$	Divide both sides by 40.
$\dfrac{b^{x-4}}{b^{x-7}} = \dfrac{4.7}{40}$	Divide both sides by b^{x-7}.
$b^{(x-4)-(x-7)} = 0.1175$	Use the quotient property of exponents.
$b^3 = 0.1175$	Combine like terms in the exponent.
$(b^3)^{1/3} = (0.1175)^{1/3}$	Raise both sides to the power $\frac{1}{3}$.
$b \approx 0.4898$	Approximate the value of $0.1175^{1/3}$.
$y = 40(0.4898)^{x-4}$	Substitute 0.4898 for *b* in either of the two original equations.

The exponential equation that passes through the points (4, 40) and (7, 4.7) is $y = 40(0.4898)^{x-4}$.

c. To find the loudness at 0 s, substitute $x = 0$.

$y = 40(0.4898)^{0-4} \approx 695$

The sound was approximately 695 lb/in.^{2} when the bell was struck.

Science CONNECTION

Sound is usually measured in bels, named after American inventor and educator Alexander Graham Bell (1847–1922). A decibel (dB) is one-tenth of a bel. The decibel scale measures loudness in terms of what an average human can hear. On the decibel scale, 0 dB is inaudible and 130 dB is the threshold of pain. However, sound can also be measured in terms of the pressure that the sound waves exert on a drum. In the metric system, pressure is measured in Pascals (Pa), a unit of force per square meter.

In Denver, Colorado, on October 1, 2000, a judge for Guinness World Records holds a device that measured the world record volume of fans roaring in a stadium— 127.8 dB.

In Example B, part b, note that the base of 0.4898 was an approximation for *b* found by dividing 4.7 by 40, then raising that quotient to the power $\frac{1}{3}$. You could use $\left(\frac{4.7}{40}\right)^{1/3}$ as an exact value of *b* in the exponential equation.

$$y = 40\left(\left(\frac{4.7}{40}\right)^{1/3}\right)^{x-4}$$

Using the power of a power property, you can rewrite this as

$$y = 40\left(\frac{4.7}{40}\right)^{(x-4)/3}$$

This equation indicates that the curve passes through the point (4, 40) and that it has a ratio of $\frac{4.7}{40}$ spread over 3 units (from $x = 4$ to $x = 7$) rather than over 1 unit. Dividing the exponent by 3 stretches the graph horizontally by 3 units. Using this method, you can write an equation for an exponential curve in only one step.

EXERCISES

Practice Your Skills

1. Match all expressions that are equivalent.

a. $\sqrt[5]{x^2}$ **b.** $x^{2.5}$ **c.** $\sqrt[3]{x}$ **d.** $x^{5/2}$ **e.** $x^{0.4}$

f. $\left(\frac{1}{x}\right)^{-3}$ **g.** $\left(\sqrt{x}\right)^5$ **h.** x^3 **i.** $x^{1/3}$ **j.** $x^{2/5}$

2. Identify each function as a power function, an exponential function, or neither of these. (It may be translated, stretched, or reflected.) Give a brief reason for your choice.

a. $f(x) = 17x^5$ **b.** $f(t) = t^3 + 5$ **c.** $g(v) = 200(1.03)^v$ **d.** $h(x) = 2x - 7$

e. $g(y) = 3\sqrt{y-2}$ **f.** $f(t) = t^2 + 4t + 3$ **g.** $h(t) = \frac{12}{3^t}$ **h.** $g(w) = \frac{28}{w-5}$

i. $f(y) = \frac{8}{y^4} + 1$ **j.** $g(x) = \frac{x^3+2}{1-x}$ **k.** $h(w) = \sqrt[5]{4w^3}$ **l.** $p(x) = 5(0.8)^{(x-4)/2}$

3. Rewrite each expression in the form b^n in which n is a rational exponent.

a. $\sqrt[6]{a}$ **b.** $\sqrt[10]{b^8}$ **c.** $\frac{1}{\sqrt{c}}$ **d.** $\left(\sqrt[5]{d}\right)^7$

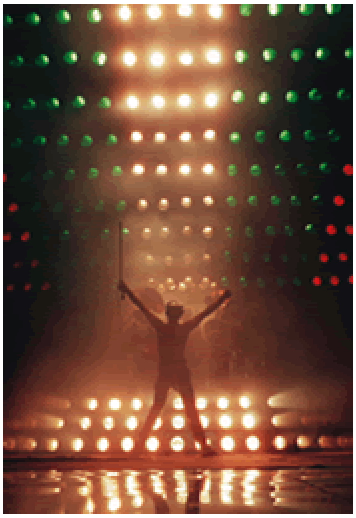
Keymath.com More Practice Your Skills

4. Solve each equation and show or explain your step(s).

a. $\sqrt[6]{a} = 4.2$ **b.** $\sqrt[10]{b^8} = 14.3$ **c.** $\frac{1}{\sqrt{c}} = 0.55$ **d.** $\left(\sqrt[5]{d}\right)^7 = 23$

Reason and Apply

5. **APPLICATION** Dan placed three colored gels over the main spotlight in the theater so that the intensity of the light on stage was 900 watts per square centimeter (W/cm^2). After he added two more gels, making a total of five over the spotlight, the intensity on stage dropped to 600 W/cm^2. What will be the intensity of the light on stage with six gels over the spotlight if you know that the intensity of light decays exponentially with the thickness of material covering it?


Colorful stage lights surround Freddie Mercury (1946–1991) of the band Queen in 1978.

6. ***Mini-Investigation*** For 6a–d, graph the equations on your calculator.

a. $y = x^{1/2}$ **b.** $y = x^{1/3}$

c. $y = x^{1/4}$ **d.** $y = x^{1/5}$

e. How do the graphs compare? What points (if any) do they have in common?

f. Predict what the graph of $y = x^{1/7}$ will look like. Verify your prediction by using your calculator.

g. What is the domain of each function? Can you explain why?

7. ***Mini-Investigation*** For 7a–d, graph the equations on your calculator.

a. $y = x^{1/4}$

b. $y = x^{2/4}$

c. $y = x^{3/4}$

d. $y = x^{4/4}$

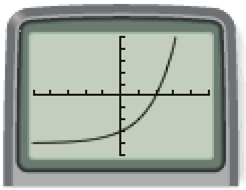
e. How do the graphs compare? What points (if any) do they have in common?

f. Predict what the graph of $y = x^{5/4}$ will look like. Verify your prediction by using your calculator.

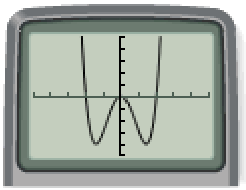
8. Compare your observations of the power functions in Exercises 6 and 7 to your previous work with exponential functions and power functions with positive integer exponents. How do the shapes of the curves compare? How do they contrast?

9. Identify each graph as an exponential function, a power function, or neither of these.

a.



b.



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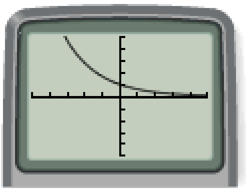

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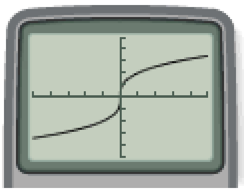
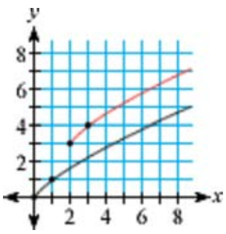
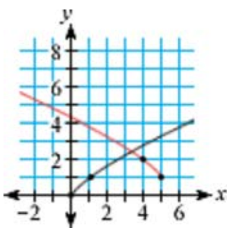


10. Each of the red curves is a transformation of the graph of the power function $y = x^{3/4}$, shown in black. Write an equation for each red curve.

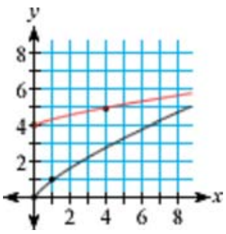
a.



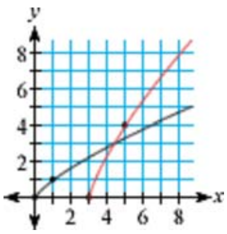
b.



c.



d.



11. Solve. Approximate answers to the nearest hundredth.

a. $9\sqrt[5]{x} + 4 = 17$ **b.** $\sqrt{5x^4} = 30$ **c.** $4\sqrt[3]{x^2} = \sqrt{35}$

12. **APPLICATION** German astronomer Johannes Kepler (1571–1630) discovered in 1619 that the mean orbital radius of a planet, measured in astronomical units (AU), is equal to the time of one complete orbit around the sun, measured in years, raised to a power of $\frac{2}{3}$.

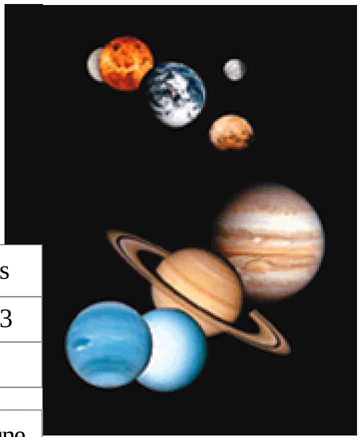
a. Venus has an orbital time of 0.615. What is its radius?

b. Saturn has a radius of 9.542 AU. How long is its orbital time?

c. Complete this table.

Planet	Mercury	Venus	Earth	Mars
Orbital radius (AU)	0.387			1.523
Orbital time (yr)		0.615	1.00	

Planet	Jupiter	Saturn	Uranus	Neptune
Orbital radius (AU)		9.542		30.086
Orbital time (yr)	11.861		84.008	


Clockwise from top left, this montage of separate planetary photos includes Mercury, Venus, Earth (and Moon), Mars, Jupiter, Saturn, Uranus, and Neptune.

13. **APPLICATION** Discovered by Irish chemist Robert Boyle (1627–1691) in 1662, Boyle's law gives the relationship between pressure and volume of gas if temperature and amount remain constant. If the volume in liters, V, of a container is increased, the pressure in millimeters of mercury (mm Hg), P, decreases. If the volume of a container is decreased, the pressure increases. One way to write this rule mathematically is $P = kV^{-1}$, where k is a constant.

a. Show that this formula is equivalent to $PV = k$.

b. If a gas occupies 12.3 L at a pressure of 40.0 mm Hg, find the constant, k.

c. What is the volume of the gas in 13b when the pressure is increased to 60.0 mm Hg?

d. If the volume of the gas in 13b is 15 L, what would the pressure be?

Science CONNECTION

Scuba divers are trained in the effects of Boyle's law. As divers ascend, water pressure decreases, and so the air in the lungs expands. It is relatively safe to make an emergency ascent from a depth of 60 ft, but you must exhale as you do so. If you were to hold your breath while ascending, the expanding oxygen in your lungs would cause your air sacs to rupture and your lungs to bleed.


A scuba diver swims below a coral reef in the Red Sea.

Review

14. Use properties of exponents to find an equivalent expression in the form ax^n.

a. $(3x^3)x^3$

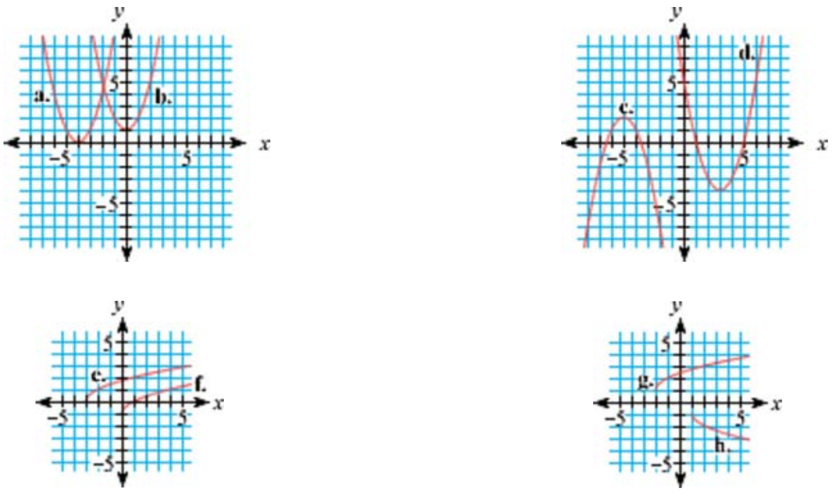
b. $(2x^3)(2x^2)^3$

c. $\frac{6x^4}{30x^5}$

d. $(4x^2)(3x^2)^3$

e. $\frac{-72x^5y^5}{-4x^3y}$ (Find an equivalent expression in the form ax^ny^m.)

15. For graphs a–h, write the equation of each graph as a transformation of $y = x^2$ or $y = \sqrt{x}$.



16. In order to qualify for the state dart championships, you must be in at least the 98th percentile of all registered dart players in the state. There are about 42,000 registered dart players. How many qualify for the championships?

17. The town of Hamlin has a growing rat population. Eight summers ago, there were 20 rat sightings, and the numbers have been increasing by about 20% each year.

a. Give a recursive formula that models the increasing rat population. Use the number of rats in the first year as u_1.

b. About how many rat sightings do you predict for this year?

c. Define variables and write an equation that models the continuous growth of the rat population.

German artist Katharina Fritsch (b 1956) designed *Rattenkönig* (*Rat-King*), giant rat sculptures with tails knotted together, based on true accounts of this rare rat pack phenomenon.

Project
POWERS OF 10

How much longer does it take 1 billion seconds to go by than 1 million seconds? How much taller are you than an ant? Are there more grains of sand on a beach than stars in the sky? You've studied how quantities change exponentially, but just how different are 10^9 and 10^{10}?

In this project you'll identify and compare objects whose *orders of magnitude* in size or number are various powers of 10. When scientists describe a quantity as having a certain order of magnitude, they look only at the power of 10, not the decimal multiplier, when the quantity is expressed in *scientific notation*. For example, 9.2×10^3 is on the order of 10^3 even though it is very close to 10^4.

Decide what you're going to measure: length, area, volume, speed, or any other quantity. Then try to find at least one object with a measurement on the order of each power of 10. Your objects can be related in some way, but they don't have to be. For instance, what is the area of your kitchen? Your house? The state you live in? The land surface of Earth? You'll probably find some powers of 10 more easily than others.

Your project should include

- A list of the object or objects you found for each power of 10, and a source or calculation for each measurement. Try to include at least 15 powers of 10.
- An explanation of any powers of 10 you couldn't find, and the largest and smallest values you found, if there are any. Don't forget negative powers.
- A visual aid or written explanation showing the different scales of your objects. If your objects are related, include an explanation of how they're related.

The film *Powers of Ten* (1977), by American designers Charles Eames (1907–1978) and Ray Eames (1912–1988), explores the vastness of the universe using the powers of 10. The film begins with a 1-meter-square image of a man in a Chicago park, which represents 10^0. Then the camera moves 10 times farther away each ten seconds until it reaches the edge of the universe, representing 10^{25}. Then the camera zooms in so that the view is ultimately an atom inside the man, representing 10^{-18}. These stills from the film show images representing 10^3 to 10^{-2}.

LESSON 5.4

Links to Resources (Keymath.com)

Applications of Exponential and Power Equations

You have seen that many equations can be solved by undoing the order of operations. In Lesson 5.2, you applied this strategy to some simple power equations. The strategy also applies for more complex power equations that arise in real-world problems.

EXAMPLE A

Rita wants to invest \$500 in a savings account so that its doubling time will be 8 years. What annual percentage rate is necessary for this to happen? (Assume the interest on the account is compounded annually.)

▶ Solution

If the doubling time is 8 yr, the initial deposit of \$500 will double to \$1000. The interest rate, r, is unknown. Write an equation and solve for r.

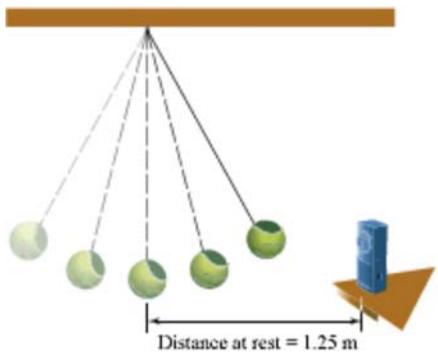
$1000 = 500(1 + r)^8$	Original equation.
$2 = (1 + r)^8$	Undo the multiplication by 500 by dividing both sides by 500.
$2^{1/8} = ((1 + r)^8)^{1/8}$	Undo the power of 8 by raising both sides to the power of $\frac{1}{8}$.
$2^{1/8} = 1 + r$	Use the properties of exponents.
$2^{1/8} - 1 = r$	Undo the addition of 1 by subtracting 1 from both sides.
$0.0905 \approx r$	Use a calculator to evaluate $2^{1/8} - 1$.

Rita will need to find an account with an annual percentage rate of approximately 9.05%.

You have also seen how you can use the point-ratio form of an exponential equation in real-world applications. The next example shows you a more complex point-ratio application.

EXAMPLE B

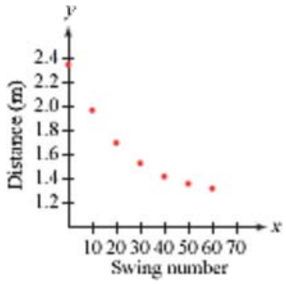
A motion sensor is used to measure the distance between it and a swinging pendulum. A table is made to record the greatest distance for every tenth swing. At rest, the pendulum hangs 1.25 m from the motion sensor. Find an equation that models the data in the table on page 262.



Swing number *x*	0	10	20	30	40	50	60
Greatest distance (m) ***y***	2.35	1.97	1.70	1.53	1.42	1.36	1.32

Solution

Plot these data. The graph shows a curved shape, so the data are not linear. As the pendulum slows, the greatest distance approaches a long-run value of 1.25 m. The pattern appears to be a shifted decreasing geometric sequence, so an exponential decay equation will provide the best model.



An exponential decay function in point-ratio form, $y = y_1 \cdot b^{x - x_1}$, will approach a long-run value of 0. Because these data approach a long-run value of 1.25, the exponential function must be translated up 1.25 units. To do so, replace y with $y - 1.25$. The coefficient, y_1, is also a y-value, so you must also replace y_1 with $y_1 - 1.25$ in order to account for the translation.

The point-ratio equation is now $y - 1.25 = (y_1 - 1.25) \cdot b^{x - x_1}$. You still need to determine the value of b, so substitute the coordinates of any point, say (10, 1.97), and solve for b.

$y - 1.25 = (y_1 - 1.25) \cdot b^{x - x_1}$	Original equation.
$y - 1.25 = (1.97 - 1.25) \cdot b^{x - 10}$	Substitute (10, 1.97) for (x_1, y_1).
$y - 1.25 = 0.72 \cdot b^{x - 10}$	Subtract within the parentheses.
$\frac{y - 1.25}{0.72} = b^{x - 10}$	Divide both sides by 0.72.
$\left(\frac{y - 1.25}{0.72}\right)^{1/(x-10)} = b$	Raise both sides to a power of $\frac{1}{x - 10}$ to solve for b.

You now have b in terms of x and y. Evaluating b for all the other data points gives these values:

Swing number *x*	0	20	30	40	50	60
Closest distance (m) ***y***	2.35	1.70	1.53	1.42	1.36	1.32
b	0.9585	0.9541	0.9539	0.9530	0.9541	0.9545

The values for b are not equal, but they are close and show no pattern. Because four out of six are close to 0.9541, it is a good choice for b, so a model for these data is $y = 1.25 + 0.72(0.9541)^{x - 10}$.

In the last example, you really hoped to see the same value for b appear six times, but when working with real measurements, the best you will usually get are close values and a small variation without a pattern.

EXERCISES

Practice Your Skills

1. Solve.

a. $x^5 = 50$ **b.** $\sqrt[3]{x} = 3.1$ **c.** $x^2 = -121$

2. Solve.

a. $x^{1/4} - 2 = 3$ **b.** $4x^7 - 6 = -2$ **c.** $3(x^{2/3} + 5) = 207$

d. $1450 = 800\left(1 + \frac{x}{12}\right)^{7.8}$ **e.** $14.2 = 222.1 \cdot x^{3.5}$

3. Rewrite each expression in the form ax^n.

a. $(27x^6)^{2/3}$ **b.** $(16x^8)^{3/4}$ **c.** $(36x^{-12})^{3/2}$

Reason and Apply

4. **APPLICATION** A sheet of translucent glass 1 mm thick is designed to reduce the intensity of light. If six sheets are placed together, then the outgoing light intensity is 50% of the incoming light intensity. What is the reduction rate of one sheet in this exponential relation?

5. Natalie performs a decay simulation using small colored candies with a letter printed on one side. She starts with 200 candies and pours them onto a plate. She removes all the candies with the letter facing up, counts the remaining candies, and then repeats the experiment using the remaining candies. Here are her data for each stage:

Stage number x	0	1	2	3	4	5	6
Candies remaining y	200	105	57	31	18	14	12

After stage 6, she checked the remaining candies and found that seven did not have a letter on either side.

a. Natalie uses the point-ratio equation, $y = y_1 \cdot b^{x - x_1}$, to model her data. What must she do to the equation to account for the seven unmarked candies? Write the equation.

b. Natalie uses the second data point, (1, 105), as (x_1, y_1). Write her equation with this point and then solve for b in terms of x and y.

c. Make a table that shows the values Natalie gets for b when substituting the other coordinates into the equation from 5b.

d. How should Natalie choose a value for b? What is her model for the data? Graph the equation with the data, and verify that the model fits reasonably well.

6. **APPLICATION** There is a power relationship between the radius of an orbit, x, and the time of one orbit, y, for the moons of Saturn. (The table at right lists 11 of Saturn's 30 moons.)

a. Make a scatter plot of these data.

b. Experiment with different values of a and b in the power equation $y = ax^b$ to find a good fit for the data. Work with a and b one at a time, first adjusting one and then the other until you have a good fit. Write a statement describing how well $y = ax^b$ fits the data.

c. Use your model to find the orbital radius of Titan, which has an orbit time of 15.945 days.

d. Find the orbital time for Phoebe, which has an orbit radius of 12,952,000 km.

Moons of Saturn

Moon	Radius (100,000 km)	Orbital time (d)
Atlas	1.3767	0.602
Prometheus	1.3935	0.613
Pandora	1.4170	0.629
Epimetheus	1.5142	0.694
Janus	1.5147	0.695
Mimas	1.8552	0.942
Enceladus	2.3802	1.370
Tethys	2.9466	1.888
Dione	3.7740	2.737
Helene	3.7740	2.737
Rhea	5.2704	4.518

(*www.solarsystem.nasa.gov*)

Science CONNECTION

In the year 2000, 12 new moons that orbit Saturn were discovered. Of the 18 previously known moons, 17 were in a regular orbit— orbiting in the same direction as the rotation of Saturn, and approximately around its equator. The 12 new moons were in irregular orbits, making them more difficult to find. Scientists theorize that these moons may have originally been 3 or 4 moons in regular orbit that collided with each other or with asteroids.

These images of Saturn's system are a compilation of photos taken by the *Voyager* I spacecraft in 1980.

7. **APPLICATION** The relationship between the weight in tons, W, and the length in feet, L, of a sperm whale is given by the formula $W = 0.000137L^{3.18}$.

a. An average sperm whale is 62 ft long. What is its weight?

b. How long would a sperm whale be if it weighed 75 tons?

8. **APPLICATION** In order to estimate the height of an *Ailanthus altissima* tree, botanists have developed the formula $h = \frac{5}{3}d^{0.8}$, where h is the height in meters and d is the diameter in centimeters.

a. If the height of an *Ailanthus altissima* tree is 18 m, find the diameter.

b. If the circumference of an *Ailanthus altissima* tree is 87 cm, estimate its height.

Science CONNECTION

Allometry is the study of size relationships between different features of an organism as a consequence of growth. Such relationships might involve weight versus length, height of tree versus diameter, or amount of fat versus body mass. Many characteristics vary greatly among different species, but within a species there may be a fairly consistent relationship or growth pattern. The study of these relationships produces mathematical models that scientists use to estimate one measurement of an organism based on another.

9. **APPLICATION** Fat reserves in birds are related to body mass by the formula $F = 0.033 \cdot M^{1.5}$, where F represents the mass in grams of the fat reserves and M represents the total body mass in grams.

a. How many grams of fat reserves would you expect in a 15 g warbler?

b. What percent of this warbler's body mass is fat?

10. According to the consumer price index in July 2002, the average cost of a gallon of whole milk was $2.74. If the July 2002 rate of inflation continued, it would cost $3.41 in the year 2024. What was the rate of inflation in July 2002?

Review

11. **APPLICATION** A sample of radioactive material has been decaying for 5 years. Three years ago, there were 6.0 g of material left. Now 5.2 g are left.

a. What is the rate of decay?

b. How much radioactive material was initially in the sample?

c. Find an equation to model the decay.

d. How much radioactive material will be left after 50 years (45 years from now)?

e. What is the half-life of this radioactive material?

12. In his geography class, Juan makes a conjecture that more people live in cities that are warm (above 50°F) in the winter than live in cities that are cold (below 32°F). In order to test his conjecture, he collects the mean temperatures for January of the 25 largest U.S. cities. These cities contained about 12% of the U.S. population in 2000.

a. Construct a box plot of these data.

b. List the five-number summary.

c. What are the range and the interquartile range for these data?

d. Do the data support Juan's conjecture? Explain your reasoning.

31.8°, 56.0°, 21.4°, 51.4°, 31.2°, 52.3°, 56.8°, 44.0°, 50.4°, 23.4°, 26.0°, 48.5°, 53.2°, 27.1°, 49.1°, 32.7°, 39.6°, 18.7°, 29.6°, 35.2° 37.1°, 44.2°, 39.1°, 29.5°, 40.5°

(*Time Almanac 2002*)

13. You have solved many systems of two equations with two variables. Use the same techniques to solve this system of three equations with three variables.

$$\begin{cases} 2x + y + 4z = 4 \\ x + y + z = \frac{1}{4} \\ -3x - 7y + 2z = 5 \end{cases}$$

IMPROVING YOUR REASONING SKILLS

Cryptic Clue

Lieutenant Bolombo found this cryptic message containing a clue about where the stolen money was hidden: $\sqrt[\frac{1}{3}]{\text{cin}}$ nati.

Where should the lieutenant look?

LESSON 5.5

Building Inverses of Functions

Success is more a function of consistent common sense than it is of genius.

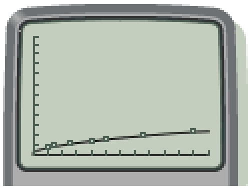
AN WANG

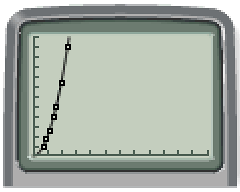
Gloria and Keith are sharing their graphs for the same set of data.

"I know my graph is right!" exclaims Gloria. "I've checked and rechecked it. Yours must be wrong, Keith."

Keith disagrees. "I've entered these data into my calculator too, and I made sure I entered the correct numbers."

The graphs are pictured below. Can you explain what is happening?





This lesson is about the **inverse** of a function—where the independent variable is exchanged with the dependent variable. Look again at Gloria's and Keith's graphs. If they labeled the axes, they might see that the only difference is their choice of independent variables. In some real-world situations, it makes sense for either of two related variables to be used as the independent variable. In the investigation you will find some equations for some inverses, and then discover how they relate to the original function.

Investigation

The Inverse

Consider the following functions.

i. $f(x) = 6 + 3x$

ii. $f(x) = \sqrt{x+4} - 3$

iii. $f(x) = (x-2)^2 - 5$

iv. $f(x) = 2 + \sqrt{x+5}$

v. $f(x) = \frac{1}{3}(x-6)$

vi. $f(x) = \sqrt[3]{5x}$

Part 1

For each of the functions above, follow Steps 1– 4.

Step 1 Using your calculator, graph the function and then draw its inverse [► See Calculator Note 5B to find out how to draw the inverse of a function. ◄]. Sketch both on your paper.

Step 2 Approximate the coordinates of at least three points on the inverse.

Step 3 Find a function (or functions) to fit the inverse. Check your response by adding this graph to your calculator to see if it matches the inverse.

Step 4 Record the equations of your function and its inverse in a table on your paper.

Part 2

Step 5 Study the sketches you made of functions and their inverses. What observations can you make about the graphs of a function and its inverse?

Step 6 Look at the graphs and equations of the pair i and v and of the pair iii and iv. What observations can you make about these pairs?

Step 7 After studying the equations you wrote for the functions and their inverses, describe how you could find the equation of an inverse of a function without looking at its graph.

EXAMPLE A

A 589 mi flight from Washington, D.C., to Chicago took 118 min. A flight of 1452 mi from Washington, D.C., to Denver took 222 min. Model this relationship both as (*time, distance*) data and as (*distance, time*) data. If a flight from Washington, D.C., to Seattle takes 323 min, what is the distance traveled? If the distance between Washington, D.C., and Miami is 910 mi, how long will it take to fly from one of these two cities to the other?

Solution

If you know the time traveled and want to find the distance, then time is the independent variable, and the points known are (118, 589) and (222, 1452). The slope is $\frac{1452-589}{222-118}$, or approximately 8.3 mi/min. Using the first point to write an equation in point-slope form, you get $d = 589 + \frac{863}{104}(t - 118)$.

To find the distance between Washington, D.C., and Seattle, substitute 323 for *time:*

$$d = 589 + \frac{863}{104}(323 - 118) \approx 2290.106$$

The distance is approximately 2290 mi.

If you know distance and want to find time, then distance is the independent variable. The two points then are (589, 118) and (1452, 222). This makes the slope $\frac{222-118}{1452-589}$, or approximately 0.12 min/mi. Using the first point again, the equation for time is $t = 118 + \frac{104}{863}(d - 589)$.

To find the time of a flight from Washington, D.C., to Miami, substitute 910 for *distance:*

$$t = 118 + \frac{104}{863}(910 - 589) \approx 156.684$$

The flight will take approximately 157 min.

You can also use the first equation for *distance* and solve for *t* to get the second equation, for *time*.

$d = 589 + \frac{863}{104}(t - 118)$	First equation.
$d - 589 = \frac{863}{104}(t - 118)$	Subtract 589 from both sides.
$\frac{104}{863}(d - 589) = (t - 118)$	Multiply both sides by $\frac{104}{863}$.
$118 + \frac{104}{863}(d - 589) = t$	Add 118 to both sides.

These two equations are inverses of each other. That is, the independent and dependent variables have been switched. Graph the two equations on your calculator. What do you notice?

In the investigation you may have noticed that the inverse of a function is not necessarily a function. Recall from Chapter 4 that any set of points is called a relation. A relation may or may not be a function.

Inverse of a Relation

You get the **inverse** of a relation by exchanging the *x*- and *y*-coordinates of all points or exchanging the *x*- and *y*-variables in an equation.

When an equation and its inverse are *both* functions, it is called a **one-to-one function.** How can you tell if a function is one-to-one?

The inverse of a one-to-one function $f(x)$ is written as $f^{-1}(x)$. Note that this notation is similar to the notation for an exponent of –1, but $f^{-1}(x)$ refers to the inverse function, not an exponent.

EXAMPLE B

Find the composition of this function with its inverse.

$$f(x) = 4 - 3x$$

Solution

The first step is to find the inverse. Exchange the independent and dependent variables. Then, solve for the new dependent variable.

$x = 4 - 3y$	Exchange *x* and *y*.
$x - 4 = -3y$	Subtract 4 from both sides.
$\frac{x-4}{-3} = y$	Divide by –3.
$f^{-1}(x) = \frac{x-4}{-3}$	Write in function notation.

The next step is to form the composition of the two functions.

$$f\left(f^{-1}(x)\right) = 4 - 3\left(\frac{x-4}{-3}\right)$$ Substitute $f^{-1}(x)$ for x in $f(x)$.

Let's see what happens when you distribute and remove some parentheses.

$$f\left(f^{-1}(x)\right) = 4 + (x - 4)$$

$$f\left(f^{-1}(x)\right) = x$$

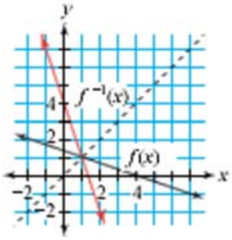
What if you had found $f^{-1}\left(f(x)\right)$ instead of $f\left(f^{-1}(x)\right)$?

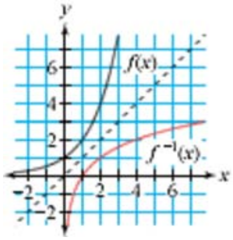
$$f^{-1}\left(f(x)\right) = \frac{(4-3x)-4}{-3}$$ Substitute $f(x)$ for x in $f^{-1}(x)$.

$$f^{-1}\left(f(x)\right) = f\left(f^{-1}(x)\right)$$ Combine like terms in the numerator.

$$f^{-1}\left(f(x)\right) = x$$ Divide.

When you take the composition of a function and its inverse, you get x. How does the graph of $y = x$ relate to the graphs of a function and its inverse? Look carefully at the graphs below to see the relationship between a function and its inverse.





A turntable DJ, like DJ Spooky shown here, applies special effects and mixing techniques to alter an original source of music. If you consider the original record to be one function and the effects or a second record to be another function, the music that the DJ creates is a composition of functions.

EXERCISES

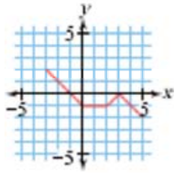
Practice Your Skills

1. A function $f(x)$ contains the points (–2, –3), (0, –1), (2, 2), and (4, 6). Give the points known to be in the inverse of $f(x)$.

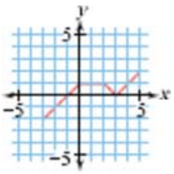
2. Given $g(t) = 5 + 2t$, find each value.

a. $g(2)$ **b.** $g^{-1}(9)$ **c.** $g^{-1}(20)$

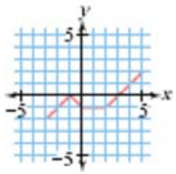
3. Which graph below represents the inverse of the relation shown in the graph at right? Explain how you know.



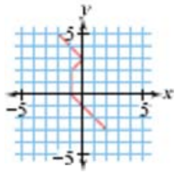
a.



b.



c.



4. Match each function with its inverse.

a. $y = 6 - 2x$ **b.** $y = 2 - \frac{6}{x}$ **c.** $y = -6(x - 2)$ **d.** $y = \frac{-6}{x-2}$

e. $y = \frac{-1}{2}(x - 6)$ **f.** $y = \frac{2}{x-6}$ **g.** $y = 2 - \frac{1}{6}x$ **h.** $y = 6 + \frac{2}{x}$

Reason and Apply

5. Given the functions $f(x) = -4 + 0.5(x - 3)^2$ and $g(x) = 3 + \sqrt{2(x+4)}$:

a. Find $f(7)$ and $g(4)$.
b. What does this imply?
c. Find $f(1)$ and $g(-2)$.
d. What does this imply?
e. Over what domain are f and g inverse functions?

6. Given $f(x) = 4 + (x - 2)^{3/5}$:

a. Solve for x when $f(x) = 12$.
b. Find $f^{-1}(x)$ symbolically.
c. How are solving for x and finding an inverse alike? How are they different?

7. Consider the function $f(x) = 4 + (x - 2)^{3/5}$ given in Exercise 6.

a. Graph $y = f(x)$ and use your calculator to draw its inverse.
b. Graph the inverse function you found in Exercise 6b. How does it compare to the inverse drawn by your calculator?
c. How can you determine whether your answer to Exercise 6b is correct?

8. Write each function using $f(x)$ notation, then find its inverse. If the inverse is a function, write it using $f^{-1}(x)$ notation.

a. $y = 2x - 3$ **b.** $3x + 2y = 4$ **c.** $x^2 + 2y = 3$

9. For each function in 9a and b, find the value of the expressions in i to iv.

a. $f(x) = 6.34x - 140$
b. $f(x) = 1.8x + 32$

i. $f^{-1}(x)$
ii. $f(f^{-1}(15.75))$
iii. $f^{-1}(f(15.75))$
iv. $f(f^{-1}(x))$ and $f^{-1}(f(x))$

Note that the equation in 9b will convert temperatures in °C to temperatures in °F. You will use either this function or its inverse in Exercise 10.

10. The data in the table describe the relationship between altitude and air temperature.

Feet	Meters	°F	°C
1,000	300	56	13
5,000	1,500	41	5
10,000	3,000	23	– 5
15,000	4,500	5	– 15
20,000	6,000	– 15	– 26
30,000	9,000	– 47	– 44
36,087	10,826	– 69	– 56

A caribou stands in front of Mt. McKinley in Denali National Park, Alaska. Mt. McKinley reaches an altitude of 6194 m above sea level.

a. Write a best-fit equation for $f(x)$ that describes the relationship (*altitude in meters, temperature in °C*). Use at least three decimal places in your answer.

b. Use your results from 10a to write the equation for $f^{-1}(x)$.

c. Write a best-fit equation for $g(x)$, describing (*altitude in feet, temperature in °F*).

d. Use your results from 10c to write an equation for $g^{-1}(x)$.

e. What would the temperature in °F be at the summit of Mount McKinley, which is 6194 m high?

f. Write a composition of functions that would also provide the answer for 10e. (You will use $f(x)$ or its inverse, $f^{-1}(x)$, from Exercise 9b.)

11. **APPLICATION** On Celsius's original scale, freezing corresponded to 100° and boiling corresponded to 0°.

a. Write a formula that converts a temperature given by today's Celsius scale into the scale that Celsius invented.

b. Explain how you would convert a temperature given in degrees Fahrenheit into a temperature on the original scale that Celsius invented.

History CONNECTION

Anders Celsius (1701–1744) was a Swedish astronomer. He created a thermometric scale using the freezing and boiling temperatures of water as reference points, on which freezing corresponded to 100° and boiling to 0°. His colleagues at the Uppsala Observatory reversed his scale five years later, giving us the current version. Thermometers with this scale were known as "Swedish thermometers" until the 1800s when people began referring to them as "Celsius thermometers."

12. Here is a paper your friend turned in for a recent quiz in her mathematics class:

If it is a four-point quiz, what is your friend's score? For each incorrect answer, provide the correct answer and explain it so that next time your friend will get it right!

QUIZ

1. Rewrite x^{-1}.
   Answer: $\frac{1}{x}$
2. What does $f^{-1}(x)$ mean?
   Answer: $\frac{1}{f(x)}$
3. Rewrite $9^{-1/5}$.
   Answer: $\frac{1}{9^5}$
4. What number is 0^0 equal to?
   Answer: 0

13. In looking over his water utility bills for the past year, Mr. Aviles saw that he was charged a basic monthly fee of \$7.18, and \$3.98 per thousand gallons (gal) used.

a. Write the monthly cost function in terms of the number of thousands of gallons used.
b. What is his monthly bill if he uses 8000 gal of water?
c. Write a function for the number of thousands of gallons used in terms of the cost.
d. If his monthly bill is \$54.94, how many gallons of water did he use?
e. Show that the functions from 13a and c are inverses.
f. Mr. Aviles decides to fix his leaky faucets. He calculates that he is wasting 50 gal/d. About how much money will he save on his monthly bill?
g. A gallon is 231 cubic inches. Find the dimensions of a rectangular container that will hold the contents of the water Mr. Aviles saves in a month.

Environmental CONNECTION

Although about two-thirds of the world is covered with water, only 1% of the Earth's water is available for drinking water. Many freshwater sources are becoming increasingly polluted or are being affected by changing weather patterns. Increasing population and consumption and the building of dams and reservoirs are damaging ecosystems around the world. Conservation of water is critical.

Leaks account for nearly 12% of the average household's annual water consumption. About one in five toilets leaks at any given time, and that can waste more than 50 gallons each day. Dripping sinks add up fast too. A faucet that leaks one drop per second can waste 30 gallons each day.

Review

14. Rewrite the expression $125^{2/3}$ in as many different ways as you can.

15. Find an exponential function that contains the points (2, 12.6) and (5, 42.525).

16. Solve by rewriting with the same base.

a. $4^x = 8^3$ **b.** $3^{4x+1} = 9^x$ **c.** $2^{x-3} = \left(\frac{1}{4}\right)^x$

17. Give the equations of two different parabolas with vertex (3, 2) passing through the point (4, 5).

18. Solve this system of equations.

$$\begin{cases} -x + 3y - z = 4 \\ 2z = x + y \\ 2.2y + 2.2z = 2.2 \end{cases}$$

LESSON

5.6

If all art aspires to the condition of music, all the sciences aspire to the condition of mathematics.

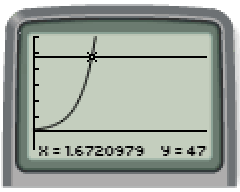
GEORGE SANTAYANA

Logarithmic Functions

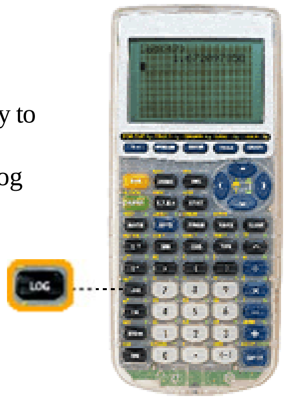
You can model many phenomena with exponential functions. You have already used several methods to solve for x when it is contained in an exponent. You've learned that in special, rare occasions, it is possible to solve by finding a common base. For example, finding the value of x that makes each of these equations true is straightforward because of your experience with the properties of exponents.

$10^x = 1000 \qquad 3^x = 81 \qquad 4^x = \frac{1}{16}$

Solving the equation $10^x = 47$ isn't as straightforward because you may not know how to write 47 as a power of 10. You can, however, solve this equation by graphing $y = 10^x$ and $y = 47$ and finding the intersection—the solution to the system and the solution to $10^x = 47$. Take a minute to verify that $10^{1.672} \approx 47$ is true.



In the investigation you will discover an algebraic strategy to solve for x in an exponential equation. You'll use a new function called a **logarithm,** abbreviated log. Locate the log key on your calculator before going on.



Investigation
Exponents and Logarithms

In this investigation you'll explore the connection between exponents with a base of 10, and logarithms.

Step 1 Enter the equation $Y_1 = 10^x$ into your calculator. Make a table of values for Y_1.

Step 2 Enter the equation $Y_2 = \log(10^x)$ and compare the table values for Y_1 and Y_2. What observations can you make? Try starting your table at different values (including negative values) and using different decimal increment values.

Step 3 Based on your observations in Step 2, what are the values of the following expressions? Use the table to verify your answers.

a. $\log(10^{2.5})$ **b.** $\log(10^{-3.2})$ **c.** $\log(10^{0})$ **d.** $\log(10^{x})$

Step 4 Complete the following statements..

a. If $100 = 10^2$, then $\log 100 = \underline{\ ?\ }$.

b. If $400 \approx 10^{2.6021}$, then $\log \underline{\ ?\ } \approx \underline{\ ?\ }$.

c. If $\underline{\ ?\ } \approx 10^{?}$, then $\log 500 \approx \underline{\ ?\ }$.

d. If $y = 10^x$, then $\log \underline{\ ?\ } = \underline{\ ?\ }$.

Step 5 Use logarithms to solve each equation for x. Check your answers.

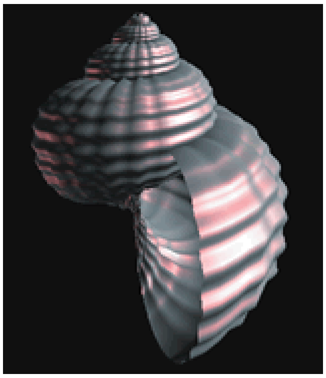
a. $300 = 10^x$ **b.** $47 = 10^x$ **c.** $0.01 = 10^x$ **d.** $y = 10^x$

Step 6 Use a friendly window with a factor of 1 to investigate the graph of $y = \log x$. Is $y = \log x$ a function? What are the domain and range of $y = \log x$?

Step 7 Graph $y = 10^x$ and draw its inverse on the same set of axes. Now graph $y = \log x$. What observations can you make?

Step 8 If $f(x) = 10^x$, then what is $f^{-1}(x)$? What is $f(f^{-1}(x))$?

The expression $\log x$ is another way of expressing x as an exponent on the base 10. Ten is the common base for logarithms, so $\log x$ is called a **common logarithm** and is shorthand for writing $\log_{10} x$. You read this as "the logarithm base 10 of x." Log x is the exponent you put on 10 to get x.



The spiral shape of this computer-generated shell was created by a logarithmic function.

EXAMPLE A

Solve $4 \cdot 10^x = 4650$.

▸ Solution

$4 \cdot 10^x = 4650$ Original equation.

$10^x = 1162.5$ Divide both sides by 4.

$x = \log_{10} 1162.5$ The logarithm base 10 of 1162.5 is the exponent you place on 10 to get 1162.5.

$x \approx 3.0654$ Use the log key on your calculator to evaluate.

The general **logarithmic function** is an exponent-producing function. The logarithm base b of x is the exponent you put on b to get x.

Definition of Logarithm

For $a > 0$ and $b > 0$, $\log_b a = x$ is equivalent to $a = b^x$.

The general logarithmic function is dependent on the base of the exponential expression. The next example demonstrates how to use logarithms to solve exponential equations when the base is not 10.

EXAMPLE B

Solve $4^x = 128$.

Solution

You know that $4^3 = 64$ and $4^4 = 256$, so if $4^x = 128$, x must be between 3 and 4. You can rewrite the equation as $x = \log_4 128$ by the definition of a logarithm. But the calculator doesn't have a built-in logarithm base 4 function. Have we hit a dead end?

One way to solve this equation is to rewrite each side of the equation $4^x = 128$ as a power with a base of 10.

$4^x = 128$	Original equation.
$(10^{0.6021})^x \approx 128$	Log 4 $\approx$ 0.6021, so $4 \approx 10^{0.6021}$.
$(10^{0.6021})^x \approx 10^{2.1072}$	Log 128 $\approx$ 2.10721, so $128 \approx 10^{2.1072}$.
$0.6021x \approx 2.1072$	Use the power of a power property of exponents and the common base property of equality.
$x \approx \dfrac{2.1072}{0.6021} \approx 3.500$	Divide both sides by 0.6021.

Recall that $x = \log_4 128$ and that 2.1072 was an approximation for log 128 and 0.6021 was an approximation for log 4. The numerator and denominator of the last step above suggest a more direct way to solve $x = \log_4 128$.

Use log 128 instead of 2.1072.

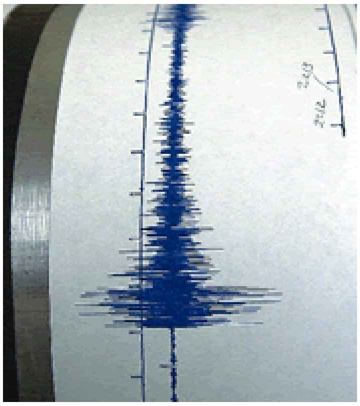
$$x = \log_4 128 = \frac{\log 128}{\log 4} = 3.5$$

Use log 4 instead of 0.6021.

The relationship above is called the logarithm change-of-base property. It enables you to solve problems involving logarithms with bases other than 10.

Logarithm Change-of-Base Property

$$\log_b a = \frac{\log a}{\log b} \text{ where } a > 0 \text{ and } b > 0$$



This relationship works because you can write any number using the inverse functions of logarithms and exponents. Doing a composition of functions that are inverses of each other produces an output value that is the same as the input. By definition, the equation $10^x = 4$ is equivalent to $x = \log 4$. Substitution from the second equation into the first equation gives you $10^{\log 4} = 4$. More generally, $10^{\log x} = x$, which means $y = 10^x$ and $y = \log x$ are inverses. This relationship allows you to rewrite any logarithmic expression with base 10. You could also choose to rewrite a logarithmic expression with any other base, so $\log_b a = \frac{\log_c a}{\log_c b}$.

Developed in 1935 by American scientist Charles F. Richter (1900–1985), the Richter scale measures the magnitude of an earthquake by taking the logarithm of the amplitude of waves recorded by a seismograph, shown at left. Because it is a logarithmic scale, each whole number increase in magnitude represents an increase in amplitude by a power of 10.

EXAMPLE C An initial deposit of \$500 is invested at 8.5% interest, compounded annually. How long will it take until the balance grows to \$800?

Solution

Let x represent the number of years the investment is held. Use the general formula for exponential growth, $y = a(1 + r)^x$.

$500(1 + 0.085)^x = 800$	Growth formula for compounding interest.
$(1.085)^x = 1.6$	Divide both sides by 500.
$x = \log_{1.085} 1.6$	Use the definition of logarithm.
$x = \dfrac{\log 1.6}{\log 1.085}$	Use the logarithm change-of-base property.
$x \approx 5.7613$	Evaluate.

It will take 6 years for the balance to grow to at least \$800.

EXERCISES

Practice Your Skills

1. Rewrite each logarithmic equation in exponential form using the definition of a logarithm.

   a. $\log 1000 = x$ **b.** $\log_5 625 = x$ **c.** $\log_7 \sqrt{7} = x$

   d. $\log_8 2 = x$ **e.** $\log_5 \frac{1}{25} = x$ **f.** $\log_6 1 = x$

2. Solve each equation in Exercise 1 for x.

3. Rewrite each exponential equation in logarithmic form using the definition of a logarithm. Then solve for x. (Give your answers rounded to four decimal places.)

   a. $10^x = 0.001$ **b.** $5^x = 100$ **c.** $35^x = 8$

   d. $0.4^x = 5$ **e.** $0.8^x = 0.03$ **f.** $17^x = 0.5$

4. Graph each equation. Write a sentence explaining how the graph compares to the graph of either $y = 10^x$ or $y = \log x$.

   a. $y = \log(x + 2)$ **b.** $y = 3 \log x$ **c.** $y = -\log x - 2$

   d. $y = 10^{x+2}$ **e.** $y = 3(10^x)$ **f.** $y = -(10^x) - 2$

Reason and Apply

5. Classify each statement as true or false. If false, change the second part to make it true.

   a. If $6^x = 12$, then $x = \log_{12} 6$. **b.** If $\log_2 5 = x$, then $5^x = 2$.

   c. If $2 \cdot 3^x = 11$, then $x = \dfrac{\log 11}{2 \log 3}$. **d.** If $x = \dfrac{\log 7}{\log 3}$, then $x = \log_7 3$.

6. The function $g(x) = 23(0.94)^x$ gives the temperature in degrees Celsius of a bowl of water x minutes after a large quantity of ice is added. After how many minutes will the water reach 5°C?

7. Assume the United States's national debt can be estimated with the model $y = 0.051517(1.1306727)^x$, where x represents the number of years since 1900 and y represents the debt in billions of dollars.

a. According to the model, when did the debt pass \$1 trillion (\$1000 billion)?
b. According to the model, what is the annual growth rate of the national debt?
c. What is the doubling time for this growth model?

8. **APPLICATION** Carbon-14 is an isotope of carbon that is formed when radiation from the sun strikes ordinary carbon dioxide in the atmosphere. Trees, which get their carbon dioxide from the air, contain small amounts of carbon-14. Once a tree is cut down, no more carbon-14 is formed, and the amount that is present begins to decay slowly. The half-life of the carbon-14 isotope is 5730 yr.

Fossilized wood can be found in Petrified Forest National Park. Some of the fossils are over 200 million years old.

a. Find an equation that models the percentage of carbon-14 in a sample of wood. (Consider that at time zero there is 100% and that at time 5730 yr there is 50%.)

b. A piece of wood contains 48.37% of its original carbon-14. According to this information, approximately how long ago did the tree that it came from die? What assumptions are you making, and why is this answer approximate?

9. **APPLICATION** Crystal looks at an old radio dial and notices that the numbers are not evenly spaced. She hypothesizes that there is an exponential relationship involved. She tunes the radio to 88.7 FM. After six "clicks" of the tuning knob, she is listening to 92.9 FM.

a. Write an exponential model in point-ratio form. Let x represent the number of clicks past 88.7 FM, and let y represent the station number.

b. Use the equation you have found to determine how many clicks Crystal should turn to get from 88.7 FM to 106.3 FM.

Review

10. Solve.

a. $(x - 2)^{2/3} = 49$ **b.** $3x^{2.4} - 5 = 16$

11. **APPLICATION** The number of railroad passengers has been increasing in the United States. The table shows railroad ridership from 1988 to 2000.

a. Plot the data and find the median-median line.

b. Calculate the residuals.

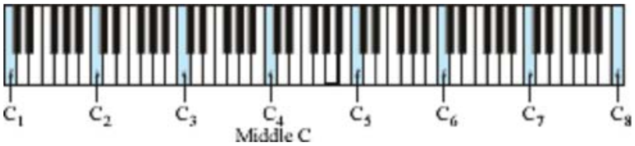
c. What is the root mean square error for this model? Explain what it means in this context.

d. If the trend continues, what is a good estimate of the ridership in 2010?

Year x	Passengers (millions) y
1988	36.9
1989	38.8
1990	40.2
1991	40.1
1992	41.6
1993	55.0
1994	60.7
1995	62.9
1996	65.6
1997	68.7
1998	75.1
1999	79.8
2000	84.1

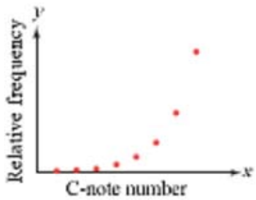
(*Time Almanac 2002*)

12. **APPLICATION** The C notes on a piano (C_1– C_8) are one octave apart. Their relative frequencies double from one C note to the next.



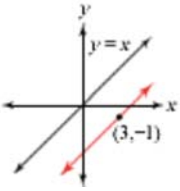
a. If the frequency of middle C (or C_4) is 261.6 cycles per second, and the frequency of C_5 is 523.2 cycles per second, find the frequencies of the other C notes.

b. Even though the frequencies of the C notes form a discrete function, you can model it using a continuous explicit function. Write a function model for these notes.

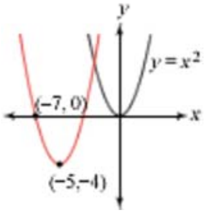


13. In each case below, use the graph and equation of the parent function to write an equation of the transformed image.

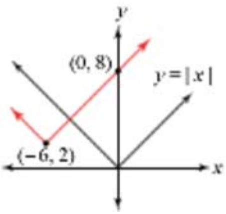
a.



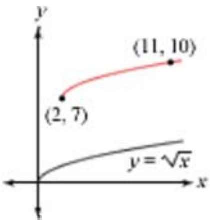
b.



c.



d.



14. A rectangle has a perimeter of 155 inches. Its length is 7 inches more than twice its width.

a. Write a system of equations using the information given.

b. Solve the system and find the rectangle's dimensions.

15. ℓ_1: $2x - 3y = 9$ ℓ_2: $2x - 3y = -1$

a. Graph ℓ_1 and ℓ_2. What is the relationship between these two lines?

b. Give the coordinates of one point, A, on ℓ_1, and two points, P and Q, on ℓ_2.

c. Describe the transformation that maps ℓ_1 onto ℓ_2 and A onto P. Write the equation of the image of ℓ_1 showing the transformation.

d. Describe the transformation that maps ℓ_1 onto ℓ_2 and A onto Q. Write the equation of the image of ℓ_1 showing the transformation.

e. Algebraically show that the equations in 15c and d are equivalent to ℓ_2.

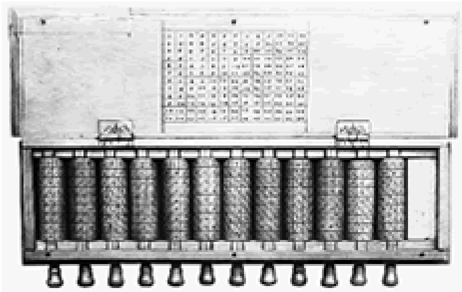
LESSON 5.7

Properties of Logarithms

Each problem that I solved became a rule which served afterwards to solve other problems.

RENÉ DESCARTES

Before machines and electronics were adapted to do multiplication, division, and raising a number to a power, scientists spent long hours doing computations by hand. Early in the 17th century, Scottish mathematician John Napier (1550–1617) discovered a method that greatly reduced the time and difficulty of these calculations, using a table of numbers that he named logarithms. As you learned in Lesson 5.6, a common logarithm is an exponent—the power of 10 that equals a number—and you already know how to use the multiplication, division, and power properties of exponents. In the next example you will discover some shortcuts and simplifications.



After inventing logarithms, John Napier designed a device for calculating with logarithms in 1617. Later called "Napier's bones," the device used multiplication tables carved on strips of wood or bone. The calculator at left has an entire set of Napier's bones carved on each spindle. You can learn more about Napier's bones and early calculating devices at www.keymath.com/DAA .

EXAMPLE

Convert numbers to logarithms to do these problems.

a. Multiply 183.47 by 19.628 without using the multiplication key on your calculator.

b. Divide 183.47 by 19.628 without using the division key on your calculator.

c. Evaluate $4.70^{2.8}$ without the exponentiation key on your calculator. (You may use the 10^x key.)

Solution

You can do parts a and b by hand. Or, you can convert to logarithms and use alternative functions.

a. $\log 183.47 \approx 2.263565$, so $10^{2.263565} \approx 183.47$.

$\log 19.628 \approx 1.292876$, so $10^{1.292876} \approx 19.628$.

$$183.47 \cdot 19.628 \approx 10^{2.263565} \cdot 10^{1.292876} = 10^{2.263565+1.292876} \approx 10^{3.556441} \approx 3601.148$$

b. $\dfrac{183.47}{19.628} \approx \dfrac{10^{2.263565}}{10^{1.292876}} = 10^{2.263565-1.292876} = 10^{0.970689} \approx 9.34736$

c. $\log 4.70 \approx 0.6721$, so $10^{0.6721} \approx 4.70$.

$$4.70^{2.8} \approx (10^{0.6721})^{2.8} = 10^{0.6721 \cdot 2.8} \approx 10^{1.8819} \approx 76.2$$

People did these calculations with a table of base-10 logarithms before there were calculators. For example, they looked up log 183.47 and log 19.628 in a table and added those together. Then they worked backward in their table to find the **antilog,** or antilogarithm, of that sum.

	0	1	2	3	4
4.0	.6021	.6031	.6042	.6053	.6064
4.1	.6128	.6138	.6149	.6160	.6170
4.2	.6232	.6243	.6253	.6263	.6274
4.3	.6335	.6345	.6355	.6365	.6375
4.4	.6435	.6444	.6454	.6464	.6474
4.5	.6532	.6542	.6551	.6561	.6571
4.6	.6628	.6637	.6646	.6656	.6665
4.7	.6721	.6730	.6739	.6749	.6758
4.8	.6812	.6821	.6830	.6839	.6848
4.9	.6902	.6911	.6920	.6928	.6937

$\log 4.70 \approx 0.6721$

	0	1	2	3	4
7.0	.8451	.8457	.8463	.8470	.8476
7.1	.8513	.8519	.8525	.8531	.8537
7.2	.8573	.8579	.8585	.8591	.8597
7.3	.8633	.8639	.8645	.8651	.8657
7.4	.8692	.8698	.8704	.8710	.8716
7.5	.8751	.8756	.8762	.8768	.8774
7.6	.8808	.8814	.8820	.8825	.8831
7.7	.8865	.8871	.8876	.8882	.8887
7.8	.8921	.8927	.8932	.8938	.8943
7.9	.8976	.8982	.8987	.8993	.8998

$\text{antilog } 0.8819 \approx 7.62$

$$\begin{array}{r} \text{antilog } 1 = 10 \\ \hline \text{antilog } 1.8819 \approx 7.62 \cdot 10 \approx 76.2 \end{array}$$

Can you see why "10 to the power" came to be called the antilog? The antilog of 3 is the same as 10^3, which equals 1000. Later, slide rules were invented to shorten this process, although logarithm tables were still used for more precise calculations. Because a logarithm is an exponent, it must have properties similar to the properties of exponents. The following investigation uses a simple slide rule to explore these properties.

Technology CONNECTION

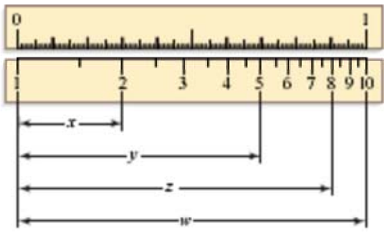
A few years after Napier's discovery, English mathematician William Oughtred (1574–1660) realized that sliding two logarithmic scales next to each other makes calculations easier, and he invented the slide rule. Over the next three centuries, many people made improvements to the slide rule, making it an indispensable tool for engineers and scientists, until computers and calculators became widely available in the 1970s. For more on the history of computational machines, see the links at www.keymath.com/DAA.

Investigation
Slide Rule

You will need

- a slide-rule worksheet
- two ordinary rulers

Work with a partner. Your slide rule is two rulers marked and scaled in unequal increments. Notice that each part of the slide rule begins with 1 (not 0).



Step 1 One part of the slide rule is shown at right, next to a decimal ruler. Use the decimal ruler to measure each of the lengths w, x, y, and z, accurate to the nearest tenth of a unit.

Step 2 Use the lengths from Step 1 to calculate these ratios and express them as decimals.

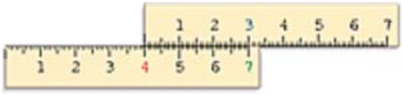
a. $\frac{x}{w}$ **b.** $\frac{y}{w}$ **c.** $\frac{y}{x}$ **d.** $\frac{z}{x}$

Step 3 Use the decimal scale and your calculations from Step 2 to find these logarithms.

a. $\log_{10} 2$ **b.** $\log_{10} 5$ **c.** $\log_2 5$ **d.** $\log_2 8$

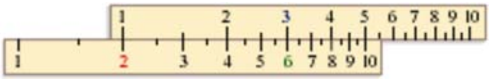
Step 4 Describe mathematically how the numbers are placed on the slide rule. Explain clearly enough so that a person reading your description could make an accurate slide rule.

Step 5 You can use two *ordinary* rulers to find the sum for an addition problem, like 4 + 3 = 7, by sliding one of the rulers along the other ruler as shown in the diagram below.



4 + 3 = 7

What happens when you use the slide rule to "add" 2 and 3 in the same way? (Instead of aligning the 0 of the ruler, you will have to match the 1 over the first number.) Write a conjecture and explain why this happens. Choose other pairs of numbers to test your conjecture.



Step 6 Explain, or sketch a picture, showing how to use the slide rule to do these calculations.

a. $3 \cdot 3$ **b.** $5 \cdot 7$ **c.** $2.5 \cdot 3.5$ **d.** $25 \cdot 35$

Step 7 Write a sentence explaining how the slide rule uses the properties of logarithms to find the value of $2.5 \cdot 3.5$.

Step 8 Using ordinary rulers, you can subtract 7 from 3 and get – 4. What happens when you use the slide rule to "subtract" 2 from 8 in the same way?

Step 9 Explain, or sketch a picture, showing how to use the slide rule to find these quotients.

a. $\frac{10}{2}$ **b.** $\frac{2.5}{3.5}$ **c.** $\frac{5}{7}$ **d.** $\frac{18}{5}$

Step 10 Write a sentence explaining how the slide rule uses the properties of logarithms to find the value of $\frac{2.5}{3.5}$.

In this chapter you have learned the properties of exponents and logarithms, summarized on the next page. You can use these properties to solve equations involving exponents. Remember to look carefully at the order of operations and then work step by step to undo each operation.

Properties of Exponents and Logarithms

Definition of Logarithm

If $x = a^m$, then $\log_a x = m$.

Product Property

$a^m \cdot a^n = a^{m+n}$ or $\log_a xy = \log_a x + \log_a y$

Quotient Property

$\frac{a^m}{a^n} = a^{m-n}$ or $\log_a \frac{x}{y} = \log_a x - \log_a y$

Power Property

$\log_a x^n = n \log_a x$

Power of a Power Property

$(a^m)^n = a^{mn}$

Power of a Product Property

$(ab)^m = a^m b^m$

Power of a Quotient Property

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Change-of-Base Property

$\log_a x = \frac{\log_b x}{\log_b a}$

Definition of Rational Exponents

$a^{m/n} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$

Definition of Negative Exponents

$a^{-n} = \frac{1}{a^n}$ or $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

EXERCISES

Practice Your Skills

1. Change the form of each expression below using properties of logarithms or exponents, without looking back in the book. Name each property or definition you use.

a. g^{h+k} **b.** $\log s + \log t$ **c.** $\frac{f^w}{f^v}$ **d.** $\log \frac{h}{k}$ **e.** $(j^s)^t$

f. $\log b^g$ **g.** $\sqrt[n]{k^m}$ **h.** $\frac{\log_s t}{\log_s u}$ **i.** $w^t w^s$ **j.** p^{-h}

In Exercises 2–4, find the missing values. Then answer the questions to learn about some properties of logarithms.

2. ***Mini-Investigation*** Find the values of 2a–f.

a. $a = \log 18$ **b.** $b = \log 71$ **c.** $c = a + b$

d. $d = \text{antilog } c$ **e.** $e = 18 \cdot 71$ **f.** $f = \log(18 \cdot 71)$

g. Describe any patterns you see.

h. Complete the statement "log a + log b = __?__."

i. The property you discovered is the product property of logarithms. Explain why this property works. (*Hint*: Name another mathematical situation in which you add when multiplying.)

3. ***Mini-Investigation*** Find the values of 3a–f.

a. $a = \log 58$ **b.** $b = \log 22$ **c.** $c = a - b$

d. $d = \text{antilog } c$ **e.** $e = \frac{58}{22}$ **f.** $f = \log\left(\frac{58}{22}\right)$

g. Describe any patterns you see.

h. Complete the statement "log a – log b = __?__."

i. The property you discovered is the quotient property of logarithms. Explain why this property works. (*Hint:* Name another mathematical situation in which you subtract when dividing.)

4. ***Mini-Investigation*** Find the values of 4a–d.

a. $a = \log 13$ **b.** $b = 3.6a$ **c.** $c = \text{antilog } b$ **d.** $d = 13^{3.6}$

e. Compare the values you found. Describe any relationship you see.

f. Complete the statement "$\log a^b$ = __?__."

g. This property is called the power property of logarithms. How would you write $\log\sqrt{a}$ in terms of log a?

5. Determine whether each equation is true or false. If false, rewrite one side of the equation to make it true. Check your answer on your calculator.

a. $\log 3 + \log 7 = \log 21$ **b.** $\log 5 + \log 3 = \log 8$ **c.** $\log 16 = 4 \log 2$

d. $\log 5 - \log 2 = \log 2.5$ **e.** $\log 9 - \log 3 = \log 6$ **f.** $\log\sqrt{7} = \log\frac{7}{2}$

g. $\log 35 = 5 \log 7$ **h.** $\log\frac{1}{4} = -\log 4$ **i.** $\frac{\log 3}{\log 4} = \log\frac{3}{4}$

j. $\log 64 = 1.5 \log 16$

Reason and Apply

6. **APPLICATION** The half-life of carbon-14, which is used in dating archaeological finds, is 5730 yr.

a. Assume that 100% of the carbon-14 is present at time 0 yr, or $x = 0$. Write the equation that expresses the percentage of carbon-14 remaining as a function of time. (This should be the same equation you found in Lesson 5.6, Exercise 8a.)

b. Suppose some bone fragments have 25% of their carbon-14 remaining. What is the approximate age of the bones?

c. In the movie *Raiders of the Lost Ark* (1981), a piece of the Ark of the Covenant found by Indiana Jones contained 62.45% of its carbon-14. What year would this indicate that the ark was constructed in?

d. Coal is formed from trees that lived about 100 million years ago. Could carbon-14 dating be used to determine the age of a lump of coal? Explain your answer.

7. **APPLICATION** This table lists the consecutive notes from middle C to the next C note. This scale is called a chromatic scale and it increases in 12 steps, called half-tones. The frequencies measured in cycles per second, or hertz (Hz), associated with the consecutive notes form a geometric sequence, in which the frequency of the last C note is double the frequency of the first C note.

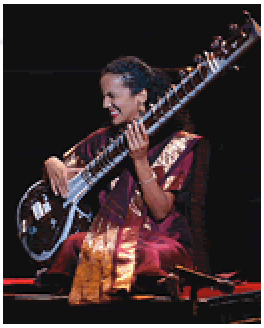
a. Find a function that will generate the frequencies.

b. Fill in the missing table values.

	Note	Frequency (Hz)
Do	C_4	261.6
	C#	
Re	D	
	D#	
Mi	E	
Fa	F	
	F#	
Sol	G	
	G#	
La	A	
	A#	
Ti	B	
Do	C_5	523.2

Music CONNECTION

If an instrument is tuned to the mathematically simple intervals that make one key sound in tune, it will sound out of tune in a different key. With some adjustments, it will be a well-tempered scale—a scale that is in tune for any key. However, not all music is based on an 8- or 12-note scale. Indian musical compositions are based on a *raga*, a structure of 5 or more notes. There are 72 *melas*, or parent scales, on which all ragas are based.



Anoushka Shankar plays the sitar in the tradition of classical Indian music.

8. Use the properties of logarithms and exponents to solve these equations.

a. $5.1^x = 247$

b. $17 + 1.25^x = 30$

c. $27(0.93^x) = 12$

d. $23 + 45(1.024^x) = 147$

9. **APPLICATION** The altitude of an airplane is calculated by measuring atmospheric pressure on the surface of the airplane. This pressure is exponentially related to the plane's height above Earth's surface. At ground level, the pressure is 14.7 pounds per square inch (abbreviated lb/in.^{2}, or psi). At an altitude of 2 mi, the pressure is reduced to 9.46 lb/in.^{2}.

a. Write an exponential equation for altitude in miles as a function of air pressure.

b. Sketch the graph of air pressure as a function of altitude. Sketch the graph of altitude as a function of air pressure. Graph your equation from 9a and its inverse to check your sketches.

c. What is the pressure at an altitude of 12,000 ft? (1 mi = 5280 ft)

d. What is the altitude of an airplane if the atmospheric pressure is 3.65 lb/in.^{2}?

Science CONNECTION

Air pressure is the weight of the atmosphere pushing down on objects within the atmosphere, including Earth itself. Air pressure decreases with increasing altitude because there is less air above you as you ascend. A barometer is an instrument that measures air pressure, usually in millibars or inches of mercury, both of which can be converted to lb/in.^{2}, which is the weight of air pressing down on each square inch of surface.

10. **APPLICATION** Carbon-11 decays at a rate of 3.5% per minute. Assume that 100% is present at time 0 min.

a. What percentage remains after 1 min?
b. Write the equation that expresses the percentage of carbon-11 remaining as a function of time.
c. What is the half-life of carbon-11?
d. Explain why carbon-11 is not used for dating archaeological finds.

Review

11. Draw the graph of a function whose inverse is not a function. Carefully describe what must be true about the graph of a function if its inverse is not a function.

12. Find an equation to fit each set of data.

a.

x	y
1	8
4	17
6	23
7	26

b.

x	y
0	2
3	54
4	162
6	1458

13. Describe how each function has been transformed from the parent function $y = 2^x$ or $y = \log x$. Then graph the function.

a. $y = -4 + 3(2)^{x-1}$

b. $y = 2 - \log\left(\frac{x}{3}\right)$

14. Answer true or false. If the statement is false, explain why or give a counterexample.

a. A grade of 86% is always better than being in the 86th percentile.
b. A mean is always greater than a median.
c. If the range of a set of data is 28, then the difference between the maximum and the mean must be 14.
d. The mean for a box plot that is skewed left is to the left of the median.

15. A driver charges \$14 per hour plus \$20 for chauffeuring if a client books directly with her. If a client books her through an agency, the agency charges 115% of what the driver charges plus \$25.

a. Write a function to model the cost of hiring the driver directly. Identify the domain and range.
b. Write a function to model what the agency charges. Identify the domain and range.
c. Give a single function that you can use to calculate the cost of using an agency to hire the driver for h hours.

LESSON 5.8

Applications of Logarithms

Drowning problems in an ocean of information is not the same as solving them.

RAY E. BROWN

In this lesson you will explore applications of the techniques and properties you discovered in the previous lesson. You can use logarithms to rewrite and solve problems involving exponential and power functions that relate to the natural world as well as to life decisions. You will be better able to interpret information about investing money, borrowing money, disposing of nuclear and other toxic waste, interpreting chemical reaction rates, and managing natural esources if you have a good understanding of these functions and problem-solving techniques.

The pH scale is a logarithmic scale. A pH of 7 is neutral. A pH reading below 7 indicates an acid, and each whole-number decrease increases acidity by a power of 10. A pH above 7 indicates an alkaline, or base, and each whole-number increase increases alkalinity by a power of 10.

EXAMPLE A

Recall the pendulum example from Lesson 5.4. The equation $y = 1.25 + 0.72(0.954)^{x-10}$ gave the greatest distance from a motion sensor for each swing of the pendulum based on the number of the swing. Use this equation to find the swing number when the greatest distance was closest to 1.47 m. Explain each step.

▸ Solution

$y = 1.25 + 0.72(0.954)^{x-10}$	Original equation.
$1.25 + 0.72(0.954)^{x-10} = 1.47$	Substitute 1.47 for *y*.
$0.72(0.954)^{x-10} = 0.22$	Subtract 1.25 from both sides.
$(0.954)^{x-10} = \frac{0.22}{0.72} \approx 0.3056$	Divide both sides by 0.72.
$\log\left((0.954)^{x-10}\right) \approx \log(0.3056)$	Take the logarithm of both sides.
$(x-10)\cdot\log(0.954) \approx \log(0.3056)$	Use the power property of logarithms.
$-0.02045(x-10) \approx -0.51485$	Evaluate the logarithms.
$x - 10 \approx \frac{-0.51485}{-0.02045} \approx 25.18$	Divide both sides by -0.02045.
$x \approx 35.18$	Add 10 to both sides.

On the 35th swing, the pendulum will be closest to 1.47 m from the motion sensor.

As you can do with other operations on equations, you can take the logarithm of both sides, as long as the value of each side is known to be positive. Recall that the domain of $y = \log x$ is $x > 0$, so you cannot find the logarithm of a negative number or zero. In Example A, you knew that both sides were equal to the positive number 0.3056 before you took the logarithm of each side.

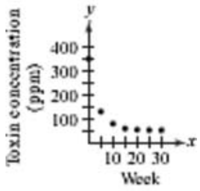
It is sometimes difficult to determine whether a relationship is logarithmic, exponential, or neither. You can use a technique called **curve straightening** to help you decide. After completing steps to straighten a curve, all you have to decide is whether or not the new graph is linear.

EXAMPLE B

Eva convinced the mill workers near her home to treat their wastewater before returning it to the lake. She then began to sample the lake water for toxin levels (measured in parts per million, or ppm) once every five weeks. Here are the data she collected.

Week	0	5	10	15	20	25	30
Toxin Level (ppm)	349.0	130.2	75.4	58.1	54.2	52.7	52.1

Eva hoped that the level would be much closer to zero after this much time. Does she have evidence that the toxin is still getting into the lake? Find an equation that models these data that she can present to the mill to prove her conclusion.



Two scientists measure toxin levels at a lake clean-up project.

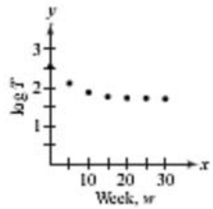
Solution

The scatter plot of the data shows exponential decay. So the model she must fit this data to is $y = k + ab^x$, where k is the toxin level the lake is dropping toward. If k is 0, then the lake will eventually be clean. If not, some toxins are still being released into the lake. If k is 0, then the general equation becomes $y = ab^x$. Take the logarithm of both sides of this equation.

$\log y = \log(ab^x)$	Take the logarithm of both sides.
$\log y = \log a + \log b^x$	Use the product property of logarithms.
$\log y = \log a + x\log b$	Use the power property of logarithms.
$\log y = c + dx$	Because $\log a$ and $\log b$ are numbers, replace them with the letters c and d for simplicity.

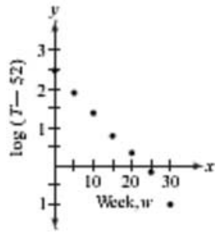
In this equation, c and d are the y-intercept and slope of a line, respectively. So, the graph of the logarithm of the toxin level over time should be linear. Let w represent the week and T represent the toxin level.

The graph of $(w, \log T)$ shown is not linear. This tells Eva that if the relationship is exponential decay, k is not 0. From the table, it appears that the toxin level may be leveling off at 52 ppm. Subtract 52 from each toxin-level measurement to test again whether $(w, \log T)$ will be linear.



w	0	5	10	15	20	25	30
T – 52	297	78.2	23.4	6.1	2.2	0.7	0.1
log(T – 52)	2.47	1.89	1.37	0.79	0.34	– 0.15	– 1.0

The graph of $(w, \log(T-52))$ does appear linear, so she can be sure that the relationship is one of exponential decay with a vertical translation of approximately 52. She now knows that the general form of this relationship is $y = 52 + ab^x$.



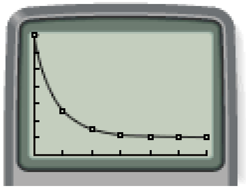
You could now use the same process as in Lesson 5.4 to solve for a and b, but the work you've done so far allows you to use an alternate method. Start by finding the median-median line for the linear data $(w, \log(T-52))$.
[See **Calculator Note 3D** to review how to find a median-median line on your calculator.]

$y = -0.110x + 2.453$	Find the median-median line.
$\log(T-52) = -0.110w + 2.453$	Substitute $\log(T-52)$ for y and w for x.
$T - 52 = 10^{-0.110w + 2.453}$	Use the definition of logarithm.
$T = 10^{-0.110w + 2.453} + 52$	Add 52 to both sides.

This is not yet in the form $y = k + ab^x$, so continue to simplify.

$T = 10^{-0.110w} \cdot 10^{2.453} + 52$	Use the product property of exponents.
$T = 10^{-0.110w} \cdot 283.79 + 52$	Evaluate $10^{2.453}$.
$T = (10^{-0.110})^w \cdot 283.79 + 52$	Use the power property of exponents.
$T = (0.776)^w \cdot 283.79 + 52$	Evaluate $10^{-0.110}$.
$T = 52 + 283.79(0.776)^w$	Reorder in the form $y = k + ab^x$.

The equation that models the amount of toxin, T, in the lake after w weeks is $T = 52 + 283.79(0.776)^w$.



If you graph this equation with the original data, you see that it fits quite well.

Investigation

Cooling

You will need

- a cup of hot water (optional)
- a temperature probe
- a data collection device
- a second temperature probe (optional)

In this investigation you will find a relationship between temperature of a cooling object and time.

Step 1 Connect a temperature probe to a data collector and set it up to collect 60 data points over 10 minutes, or 1 data point every 10 seconds. Heat the end of the probe by placing it in hot water or holding it tightly in the palm of your hand. When it is hot, set the probe on a table so that the tip is not touching anything and begin data collection. [See Calculator Note 5E.]

Step 2 Let t be the time in seconds, and let p be the temperature of the probe. While you are collecting the data, draw a sketch of what you expect the graph of (t, p) data to look like as the temperature probe cools. Label the axes and mark the scale on your graph. Did everyone in your group draw the same graph? Discuss any differences of opinion.

Step 3 Plot the data in the form (t, p) on an appropriately scaled graph. Your graph should appear to be an exponential function. Study the graph and the data, and guess the temperature limit L. You could also use a second temperature probe to measure the room temperature, L.

Step 4 Subtract this limit from your temperatures and find the logarithm of this new list. Plot data in the form $(t, \log(p - L))$. If the data are not linear, then try a different limit.

Step 5 Find the equation that models the data in Step 4, and use this to find an equation that models the (t, p) data in Step 3. Give real-world meaning to the values in the final equation.

EXERCISES

Practice Your Skills

1. Prove that these statements of equality are true. Take the logarithm of both sides, then use the properties of logarithms to re-express each side until you have two identical expressions.

a. $10^{n+p} = (10^n)(10^p)$ **b.** $\frac{10^d}{10^e} = 10^{d-e}$

2. Solve each equation. Check your answers by substituting your answer for x.

a. $800 = 10^x$ **b.** $2048 = 2^x$ **c.** $16 = 0.5^x$

d. $478 = 18.5(10^x)$ **e.** $155 = 24.0(1.89^x)$ **f.** $0.0047 = 19.1(0.21^x)$

3. Suppose you invest $3000 at 6.75% annual interest compounded monthly. How long will it take to triple your money?

Reason and Apply

4. **APPLICATION** The length of time that milk (and many other perishable substances) will stay fresh depends on the storage temperature. Suppose that milk will stay fresh for 146 hours in a refrigerator at 4°C. Milk that is left out in the kitchen at 22°C will keep for only 42 hours. Because bacteria grow exponentially, you can assume that freshness decays exponentially.

a. Write an equation that expresses the number of hours, h, that milk will keep in terms of the temperature, T.

b. Use your equation to predict how long milk will keep at 30°C and at 16°C.

c. If a container of milk soured after 147 hours, what was the temperature at which it was stored?

d. Graph the relationship between hours and temperature, using your equation from 4a and the five data points you have found.

e. What is a realistic domain for this relationship? Why?

Science CONNECTION

In 1860, French chemist Louis Pasteur (1822–1895) developed a method of killing bacteria in fluids. Today the process, called pasteurization, is routinely used on milk. It involves heating raw milk not quite to its boiling point, which would affect its taste and nutritional value, but to 63°C (145°F) for 30 min or 72°C (161°F) for 15 s. This kills most, but not all, harmful bacteria. Refrigerating milk slows the growth of the remaining bacteria, but eventually the milk will spoil, when there are too many bacteria for it to be healthful. The bacteria in milk change the lactose to lactic acid, which smells and tastes bad to humans.


Louis Pasteur

5. The equation $f(x) = \frac{12000}{1 + 499(1.09)^{-x}}$ gives the total sales x days after the release of a new video game. Find each value and give a real-world meaning.

a. $f(20)$

b. $f(80)$

c. x when $f(x) = 6000$

d. Show the steps to solve 5c symbolically.

e. Graph the equation in a window large enough for you to see the overall behavior of the curve. Use your graph to describe how the number of games sold each day changes. Does this model seem reasonable?

6. **APPLICATION** The intensity of sound, D, measured in decibels (dB) is given by the formula

$$D = 10\log\left(\frac{I}{10^{-16}}\right)$$

where I is the power of the sound in watts per square centimeter (W/cm^2) and 10^{-16} W/cm^2 is the power of sound just below the threshold of hearing.

a. Find the number of decibels of a 10^{-13} W/cm^2 whisper.

b. Find the number of decibels in a normal conversation of $3.16 \cdot 10^{-10}$ W/cm^2.

c. Find the power of the sound (in W/cm^2) experienced by the orchestra members seated in front of the brass section, measured at 107 dB.

d. How many times more powerful is a sound of 47 dB than a sound of 42 dB?

7. **APPLICATION** The table gives the loudness of spoken words, measured at the source, and the maximum distance at which another person can recognize the speech. Find an equation that expresses the maximum distance as a function of loudness.

Loudness (dB)	Distance (m)
0.5	0.1
3.2	16.0
5.3	20.4
16.8	30.5
35.8	37.0
84.2	44.5
120.0	47.6
170.0	50.6

a. Plot the data on your calculator and make a rough sketch on your paper.

b. Experiment to find the relationship between x and y by plotting different combinations of x, y, $\log x$, and $\log y$ until you have found the graph that best linearizes the data. Sketch this graph on your paper and label the axes with x, y, $\log x$, or $\log y$, as appropriate.

c. Find the equation of a line that fits the plot you chose in 7b. Remember that your axes did not represent x and y, so substitute $(\log x)$ or $(\log y)$ into your equation as appropriate.

d. Graph this new equation with the original data. Does it seem to be a good model?

8. A container of juice is left in a room at a temperature of 74°F. After 8 minutes, the temperature is recorded at regular intervals.

Time (min)	8	10	12	14	16	18	20	22	24	26	28	30
Temp. (°F)	35	40	45	49	52	55	57	60	61	63	64	66

a. Plot the data using an appropriate window. Make a rough sketch of this graph.

b. Find an exponential model for temperature as a function of time. (*Hint:* This curve is both reflected and translated.)

9. **APPLICATION** In clear weather, the distance you can see from a window on a plane depends on your height above Earth, as shown in the table at right.

a. Graph various combinations of x, y, $\log x$, and $\log y$ until you find a combination that linearizes the data.

b. Use your results from 9a to find a best-fit equation for data in the form (*height*, *view*) using the data in this table.

Height (m)	Viewing distance (km)
305	62
610	88
914	108
1,524	139
3,048	197
4,572	241
6,096	278
7,620	311
9,144	340
10,668	368
12,192	393

10. Quinn starts treating her pool for the season with a shock treatment of 4 gal of chlorine. Every 24 h, 15% of the chlorine evaporates. The next morning, she adds 1 qt ($\frac{1}{4}$ gal) of chlorine to the pool, and she continues to do so each morning.

a. How much chlorine is there in the pool after one day (after she adds the first daily quart of chlorine)? After two days? Write a recursive formula for this pattern.

b. Use the formula from 10a to make a table of values and sketch a graph of 20 terms. Find an explicit model that fits the data.

Review

11. Find these functions.

a. Find an exponential function that passes through the points (4, 18) and (10, 144).
b. Find a logarithmic function that passes through the points (18, 4) and (144, 10).

12. The Highland Fish Company is starting a new line of frozen fish sticks. It will cost $19,000 to set up the production line and $1.75 per pound to buy and process the fish. Highland Fish will sell the final product at a wholesale cost of $1.92 per pound.

a. Write a cost function and an income function for HFC's new venture.
b. Graph both functions on the same axes over the domain $0 \le x \le 1{,}000{,}000$.
c. How many pounds of fish sticks will HFC have to produce before it starts making a profit on the new venture?
d. How much profit can HFC expect to make on the first 500,000 pounds of fish?

13. Sketch the graph of $\left(4(x+5)\right)^2 + \left(\frac{y-8}{2}\right)^2 = 1$. Give coordinates of a few points that define the shape.

14. Solve each equation. Round to the nearest hundredth.

a. $x^5 = 3418$ **b.** $(x - 5.1)^4 = 256$ **c.** $7.3x^6 + 14.4 = 69.4$

Project

INCOME BY GENDER

The median annual incomes of year-round full-time workers in the United States, ages 25 and above, are listed in this table. Examine different relationships, such as data in the form (*time*, *men*), (*time*, *women*), (*women*, *men*), (*time*, *men* – *women*), (*time*, *men/women*), and so on. Find best-fit models for those relationships that seem meaningful. Write an article in which you interpret some of your models and make predictions about the future. Research some recent data to see if your predictions are accurate so far.

Year	Men	Women
1970	$9,521	$5,616
1972	$11,148	$6,331
1974	$12,786	$7,370
1976	$14,732	$8,728
1978	$16,882	$10,121
1980	$20,297	$12,156
1982	$22,857	$14,477
1984	$25,497	$16,169
1986	$27,335	$17,675

(*1990 Statistical Abstract of the United States*)

Your project should include

- Your article, including relevant graphs, models, and predictions.
- More recent data (remember to cite your source).
- An analysis of how well the recent data fit your predictions.

"Women constitute half the world's population, perform nearly two-thirds of its work hours, receive one-tenth of the world's income, and own less than one-hundredth of the world's property." (United Nations report, 1980)

EXPLORATION

The Number *e*

You've done problems exploring the amount of interest earned in a savings account when interest is compounded yearly, monthly, or daily. But what if interest is compounded continuously? This means that at every instant, your interest is redeposited in your account and the new interest is calculated on it. This type of continuous growth is related to a number, ***e***. This number has a value of approximately 2.71, and like π, it is a **transcendental number**—a number that has infinitely many nonrepeating digits. The logarithm function with base e, $\log_e x$, is also written as $\ln x$, and is called the **natural logarithm** function.

Activity

Continuous Growth

The Swiss mathematician Jacob Bernoulli (1654-1705) explored the following problem in 1683. Suppose you put $1 into an account that earns 100% interest per year. If the interest is compounded only once, at the end of the year you will have earned $1 in interest and your balance will be $2. What if interest is compounded more frequently? Follow the steps below to analyze this situation.

Step 1 If interest is compounded 10 times annually, how much money will be in your account at the end of one year? Remember that if 100% interest is compounded ten times, you earn 10% each time. Check your answer with another group to be sure you did this correctly.

Step 2 Predict what will happen if your money earns interest compounded continuously.

Step 3 What will be the balance of your account at the end of one year if interest is compounded 100 times? 1,000 times? 10,000 times? 1,000,000 times? How do your answers compare to your prediction in Step 2?

Step 4 Write an equation that would tell you the balance if the interest were compounded x times annually, and graph it on your calculator. Does this equation seem to be approaching one particular long-run value, or limit? If so, what is it? If not, what happens in the long run?

Step 5 Look for e on your calculator, and find its value to six decimal places. What is the relationship between this number and your answer to Step 4?

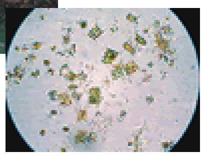
Step 6 When interest is compounded continuously, the formula $y = p\left(1 + \frac{r}{x}\right)^{xt}$ becomes $y = Pe^{rt}$, where P is the principal (the initial value of your investment), e is the natural exponential base, r is the interest rate, and t is the amount of time the investment is left in the account. Use this formula to calculate the value of $1 deposited in an account earning 100% annual interest for one year. Calculate the value if it is left in the same account for ten years.

Questions

1. The formula $y = Pe^{rt}$ is often an accurate model for growth or decay problems, because things usually grow or decay continuously, not at specific time intervals. For example, a bacteria population may double every four hours, but it is increasing throughout those four hours, not suddenly doubling in value when exactly four hours have passed. Return to any problem involving growth or decay from this chapter, and use this new formula to solve the problem. How does your answer compare with your initial answer?
2. The intensity, I, of a beam of light after passing through t cm of liquid is given by $I = Ae^{-kt}$, where A is the intensity of the light when it enters the liquid, e is the natural exponential base, and k is a constant for a particular liquid. On a field trip, a marine biology class took light readings at Deep Lake. At a depth of 50 cm, the light intensity was 80% of the light intensity at the surface.
   a. Find the value of k for Deep Lake.
   b. If the light intensity is measured as 1% of the intensity at the surface, at what depth was the reading taken?

Science CONNECTION

Phytoplankton is a generic name for a great variety of microorganisms, including algae, that live in lakes and oceans and provide the lowest step of the food chain in some ecosystems. Phytoplankton require light for photosynthesis. In water, light intensity decreases with depth. As a result, the phytoplankton production rate, which is determined by the local light intensity, decreases with depth. Light levels determine the maximum depth at which these organisms can grow. Limnologists, scientists who study inland waters, estimate this depth to be the point at which the amount of light available is reduced to 0.5–1% of the amount of light available at the lake surface.



Algae grow in abundance on this pond that holds waste water for the nearby milkhouse. At right is a microscopic image of fresh water phytoplankton.

Project

ALL ABOUT e

Mathematicians have been exploring e and calculating more digits of the decimal approximation of e since the 1600s. There are a number of procedures that can be used to calculate digits of e. Do some research at the library or on the Internet, and find at least two. Write a report explaining the formulas you find and how they are used to calculate e. Include any interesting historical facts you find on e as well.

CHAPTER 5 REVIEW

Exponential functions provide explicit and continuous equations to model geometric sequences. They can be used to model the growth of populations, the decay of radioactive substances, and other phenomena. The general form of an exponential function is $y = ab^x$, where a is the initial amount and b is the base that represents the rate of growth or decay. Because the exponent can take on all real number values, including negative numbers and fractions, it is important that you understand the meaning of these exponents. You also used the **point-ratio form** of this equation, $y = y_1 \cdot b^{x - x_1}$.

Until you read this chapter, you had no way to solve an exponential equation, other than guess-and-check. Once you defined the **inverse** of the exponential function—the **logarithmic function**—you were able to solve exponential functions symbolically. The inverse of a function is the relation you get when all values of the independent variable are exchanged with the values of the dependent variable. The graphs of a function and its inverse are reflected across the line $y = x$. The definition of the logarithmic function is that $\log_b y = x$ means that $y = b^x$. You learned that the properties of logarithms parallel those of exponents: the logarithm of a product is the sum of the logarithms, the logarithm of a quotient is the difference of the logarithms, and the logarithm of a number raised to a power is the product of the logarithm and that number. By looking at the logarithms of the x-value, or the y-value, or both values in a set of data, you can determine what type of equation will best model the data by finding which of these creates the most linear graph.

EXERCISES

1. Evaluate each expression without using a calculator. Then check your work with a calculator.

a. 4^{-2} **b.** $(-3)^{-1}$ **c.** $\left(\frac{1}{5}\right)^{-3}$ **d.** $49^{1/2}$ **e.** $64^{-1/3}$

f. $\left(\frac{9}{16}\right)^{3/2}$ **g.** -7^0 **h.** $(3)(2)^2$ **i.** $(0.6^{-2})^{-1/2}$

2. Rewrite each expression in another form.

a. $\log x + \log y$ **b.** $\log \frac{z}{y}$ **c.** $(7x^{2.1})(0.3x^{4.7})$

d. $\log w^k$ **e.** $\sqrt[5]{x}$ **f.** $\log_5 t$

3. Use the properties of exponents and logarithms to solve each equation. Confirm your answers by substituting them for x.

a. $4.7^x = 28$ **b.** $4.7^{x^2} = 2209$ **c.** $\log_x 2.9 = 1.25$ **d.** $\log_{3.1} x = 47$

e. $7x^{2.4} = 101$ **f.** $9000 = 500(1.065)^x$ **g.** $\log x = 3.771$ **h.** $\sqrt[5]{x^3} = 47$

4. Solve for x. Round your answers to the nearest thousandth.

a. $\sqrt[8]{2432} = 2x + 1$ **b.** $4x^{2.7} = 456$ **c.** $734 = 11.2(1.56)^x$

d. $f(f^{-1}(x)) = 20.2$ **e.** $147 = 12.1(1 + x)^{2.3}$ **f.** $2\sqrt{x-3} + 4.5 = 16$

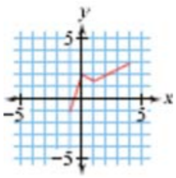
5. Once a certain medicine is in the bloodstream, its half-life is 16 h. How long (to the nearest 0.1 h) will it be before an initial 45 cm^3 of the medicine has been reduced to 8 cm^3?

6. Given $f(x) = (4x - 2)^{1/3} - 1$, find:

a. $f(2.5)$ **b.** $f^{-1}(x)$ **c.** $f^{-1}(-1)$ **d.** $f(f^{-1}(12))$

7. Find the equation of an exponential curve through the points (1, 5) and (7, 32).

8. Draw the inverse of $f(x)$, shown at right.



9. Your head gets larger as you grow. Most of the growth comes in the first few years of life, and there is very little additional growth after you reach adolescence. The estimated percentage of adult size for your head is given by the formula $y = 100 - 80(0.75)^x$, where x is your age in years and y is the percentage of the average adult size.

a. Graph this function.
b. What are the reasonable domain and range of this function?
c. Describe the transformations of the graph of $y = (0.75)^x$ that produce the graph in 9a.
d. A 2-year-old child's head is what percentage of the adult size?
e. About how old would a person be if his or her head circumference is 75% that of an average adult?

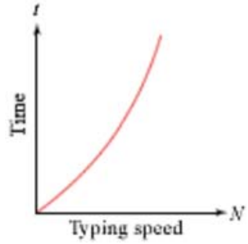
10. A new incentive plan for the Talk Alot long-distance phone company varies the cost of a call according to the formula $cost = a + b \log t$, where t represents time in minutes. When calling long distance, the cost for the first minute is \$0.50. The cost for 15 min is \$3.44.

a. Find the a-value in the equation.
b. Find the b-value in the equation.
c. What is the x-intercept of the graph of the equation? What is the real-world meaning of the x-intercept?
d. Use your equation to predict the cost of a 30-minute call.
e. If you decide you can afford only to make a \$2 call, how long can you talk?

11. **APPLICATION** A "learning curve" describes the rate at which a task can be learned. Suppose the equation

$$t = -144\log\left(1 - \frac{N}{90}\right)$$

predicts the time t (in number of short daily sessions) it will take to achieve a goal of typing N words per minute (wpm).



a. Using this equation, how long should it take someone to learn to type 40 wpm?

b. If the typical person had 47 lessons, then what speed would you expect him or her to have achieved?

c. Interpret the shape of the graph as it relates to learning time. What domain is realistic for this problem?

12. **APPLICATION** All humans start as a single cell. This cell splits into two cells, then each of those two cells splits into two cells, and so on.

a. Write a recursive formula for cell division starting with a single cell.

b. Write an explicit formula for cell division.

c. Sketch a graph to model the formulas in 12a and b.

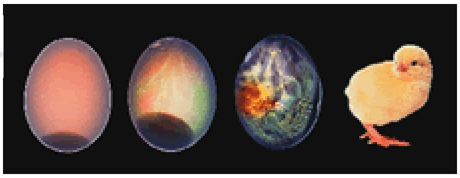
d. Describe some of the features of the graph.

e. After how many divisions were there more than 1 million cells?

f. If there are about 1 billion cells after 30 divisions, after how many divisions were there about 500 million cells?

Science CONNECTION

Embryology is the branch of biology that deals with the formation, early growth, and development of living organisms. In humans, the growth of an embryo takes about 9 months. During this time a single cell will grow into many different cell types with different shapes and functions in the body.
A similar process occurs in the embryo of any animal. Historically, chicken embryos were among the first embryos studied. A chicken embryo develops and hatches in 20–21 days. Cutting a window in the eggshell allows direct observation for the study of embryonic growth.



These X-rays show the 21-day growth of a chicken embryo to a newborn chick.

TAKE ANOTHER LOOK

1. Is $(x^{1/m})^n$ always equivalent to $(x^n)^{1/m}$? Try graphing Y1 = $(x^{1/m})^n$ and Y2 = $(x^n)^{1/m}$ for various integer values of m and n. Make sure you try positive and negative values for m and n, as well as different combinations of odd and even numbers. Check to see if the expressions are equal by inspecting the graphs and looking at table values for positive and negative values of x. Make observations about when output values are different and when output values do not exist. Make conjectures about the reasons for the occurrence of different values or no values.

2. You have learned to use the point-ratio form to find an exponential curve that fits two data points. You could also use the general exponential equation $y = ab^x$ and your knowledge of solving systems of equations to find an appropriate exponential curve. For example, you can use the general form to write two equations for the exponential curve through (4, 40) and (7, 4.7):

$40 = ab^4$

$4.7 = ab^7$

Which constant will be easier to solve for in each equation, a or b? Solve each equation for the constant you have chosen. Use substitution and the properties you have learned in this chapter to solve this system of equations. Substitute the values you find for a and b into the general exponential form to write a general equation for this function.

Find a problem in this chapter that you did using the point-ratio form, and solve it again using this new method. Did you find this method easier or more difficult to use than the point-ratio form? Are there situations in which one method might be preferable to the other?

3. As you become more familiar with a slide rule, you might discover other shortcuts. For example, here is another way to multiply 5×7:

Line up the 5 on the top scale with 10 on the bottom scale, then find the number on the top scale that is directly above 7. The answer you find is 3.5, which you then have to multiply by 10 to find the correct answer, which is 35.

Why does the shortcut work? Does it also work if you line up 7 with 10 and read off the number above 5?

Assessing What You've Learned

GIVE A PRESENTATION Give a presentation about how to fit an exponential curve to data or one of the Take Another Look activities. Prepare a poster or visual aid to explain your topic. Work with a group, or give a presentation on your own.

ORGANIZE YOUR NOTEBOOK Review your notebook to be sure it's complete and well organized. Make sure your notes include all of the properties of exponents and logarithms, including the meanings of negative and fractional exponents. Write a one-page chapter summary based on your notes.